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CHAPTER

9 Roots and Radicals

Roger has a ladder that will extend to a length of 21 feet. For the ladder to be safe to climb on, it must be placed 7 feet away from the house. The roof is 20 feet above the ground. Will the ladder be able to reach the roof safely? If not, how far up will the ladder reach? (Leave your answer rounded to one decimal place.)

9–1 ■ Principal roots

Square root

In chapter 4, quadratic equations were solved by factoring, but many quadratic equations such as

$$x^2 - 7 = 0 \quad \text{or} \quad x^2 + 3x + 1 = 0$$

will not factor over the set of rational numbers. We need to be able to solve equations that involve a squared variable. Therefore, we want a process that is the inverse of squaring a number.

In chapter 1, we discussed how to square a number. Consider the following examples:

$$\text{If } x = 3, \quad \text{then } x^2 = (3)^2 = (3)(3) = 9$$
$$\text{If } x = -3, \text{ then } x^2 = (-3)^2 = (-3)(-3) = 9$$

Reversing the process, we ask:

$$\text{If } x^2 = 9, \text{ then what number is } x \text{ equal to?}$$

This inverse operation is called *finding the square root of a number.*

Definition

For every pair of real numbers a and b, if $a^2 = b$, then a is called a square root of b.

Concept

A square root of a number is one of two equal factors of the number.

From this discussion and the definition of square root, we can see that the answer to the question we asked,

$$\text{If } x^2 = 9\text{, then what number is } x \text{ equal to?}$$

is 3 or -3 since $(3)^2 = 9$ and $(-3)^2 = 9$. To distinguish between the two square roots, we define the *principal square root* of a positive number to be positive. The $\sqrt{\ }$ symbol denotes the principal square root.
Thus, if $x = \sqrt{9}$, then $x = 3$ and we say 3 is the principal square root of 9.

$$\sqrt{9} = 3 \text{ (principal square root)}$$

The parts of the principal square root are

$$\sqrt{a}$$

Radical sign Radicand

The entire expression is called a *radical* and is read "the principal square root of a."

Example 9–1 A

Find the principal square root.

1. $\sqrt{16} = 4$, since $4 \cdot 4 = 4^2 = 16$.

2. $\sqrt{49} = 7$, since $7 \cdot 7 = 7^2 = 49$.

3. $\sqrt{25} = 5$, since $5 \cdot 5 = 5^2 = 25$.

4. $\sqrt{36} = 6$, since $6 \cdot 6 = 6^2 = 36$.

5. $\sqrt{0} = 0$, since $0 \cdot 0 = 0^2 = 0$.

Note In the examples, 0, 16, 49, 25, and 36 are called **perfect-square integers** because their square roots are integers. ■

Whenever we wish to express the negative value of the square root of a number, we use the symbol $-\sqrt{\ }$. For example, $-\sqrt{9}$ would indicate that we want the negative square root value. That is, $-\sqrt{9} = -3$.

Example 9–1 B

Find the indicated root.

1. $-\sqrt{4} = -2$

2. $-\sqrt{49} = -7$

3. $-\sqrt{25} = -5$

4. $-\sqrt{36} = -6$

▶ ***Quick check*** Find the square root. $-\sqrt{16}$ ■

Our first examples of finding the square root of a number have dealt only with perfect-square integers. We shall now try to find the $\sqrt{2}$. We could use 1.414 but when we square 1.414, we do not get 2 as an answer. We can show that no matter how many decimal places the answer is carried to, when the result is squared, it will be close to, but not equal, 2. The $\sqrt{2}$ is called an **irrational number** because it has the property that it cannot be expressed as a terminating or a repeating decimal number. Another number that is irrational is π, which is used in geometric formulas involving circles.

Whenever you work with irrational numbers in a problem, you may have to approximate the number to as many decimal places as are needed in the problem by using a calculator or a table of values.

Example 9–1 C

For the following irrational numbers, find the decimal approximation to three decimal places by using a calculator.

1. $\sqrt{3} \approx 1.732$

Note "$\approx$" is read "is approximately equal to" and is used when our answer is not exact. Square roots of integers that are not perfect squares will be irrational.

2. $-\sqrt{41} \approx -6.403$ 3. $\sqrt{50} \approx 7.071$

▶ ***Quick check*** Find the decimal approximation to three decimal places by using a calculator. $\sqrt{48}$ and $-\sqrt{56}$ ■

Not all real numbers have a rational or an irrational square root. Consider the following:

$$\sqrt{-4} = \text{what?}$$

We know that all real numbers are either positive, negative, or zero. If we square a real number, the product is never negative. Hence, there is no real number that when squared produces a negative answer. *The square root of a negative number does not exist in the set of real numbers.*

Example 9–1 D

Suppose an automotive engineer wishes to determine the diameter of the cylinder bore (D) required to produce H horsepower from N cylinders of an engine that is turning 1,000 rpm. The engineer will use the formula

$$D = \sqrt{\frac{H}{(0.4)\,N}}$$

What would be the bore diameter (in inches) required to produce 40 horsepower at 1,000 rpm from a 4-cylinder engine?

We substitute 40 for H and 4 for N.

$$D = \sqrt{\frac{(40)}{(0.4)(4)}} = \sqrt{\frac{40}{1.6}} = \sqrt{25} = 5$$

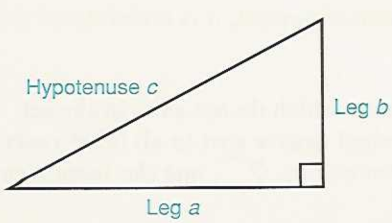
Therefore, we need a bore diameter of 5 inches. ■

Pythagorean Theorem

The following is an important property of right triangles called the **Pythagorean Theorem.**

In a right triangle, the square of the length of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the lengths of the two legs (the sides that form the right angle). If c is the length of the hypotenuse and a and b are the lengths of the legs, this property can be stated as:

$$c^2 = a^2 + b^2 \text{ or } c = \sqrt{a^2 + b^2}$$
also as
$$a^2 = c^2 - b^2 \text{ or } a = \sqrt{c^2 - b^2}$$
and as
$$b^2 = c^2 - a^2 \text{ or } b = \sqrt{c^2 - a^2}$$



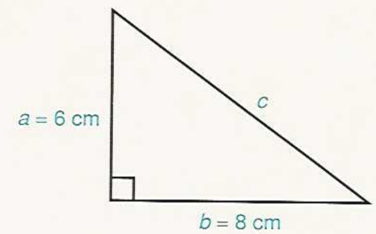
■ Example 9–1 E

Find the length of the hypotenuse of a right triangle whose legs are 6 centimeters and 8 centimeters.

We want to find c when $a = 6$ cm and $b = 8$ cm.
By the Pythagorean Theorem,

$$\begin{aligned} c &= \sqrt{a^2 + b^2} \\ &= \sqrt{6^2 + 8^2} && \text{Substitute} \\ &= \sqrt{36 + 64} && \text{Square the values} \\ &= \sqrt{100} = 10 \end{aligned}$$

Hence, $c = 10$ cm.



▶ ***Quick check*** Find the second leg of a right triangle whose hypotenuse has length 13 inches and whose first leg is 5 inches long. ■

*n*th roots

The concept of square root can be extended to find cube roots (third root of a number), fourth roots, fifth roots, and so on. A cube root is one of three equal factors of a number. The symbol that is used to express the principal cube root is $\sqrt[3]{}$. The 3 is called the **index** of the radical expression. The index denotes what root we are looking for. The principal fourth root would be indicated by $\sqrt[4]{}$. General notation for the principal nth root would be $\sqrt[n]{}$, where n is a natural number greater than 1.

The principal *n*th root

The principal nth root of number a, denoted by

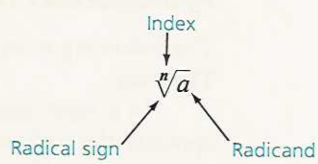
$$\sqrt[n]{a}$$

is one of n equal factors such that $\sqrt[n]{a} = b$ and

$$\overbrace{b \cdot b \cdot b \cdots b}^{n \text{ factors}} = b^n = a$$

where n *is a natural number greater than 1.*

The parts of the principal nth root are



Note If there is no index associated with a radical symbol, it is understood to be 2.

If we exclude even roots of negative numbers, which do not exist in the set of real numbers, we can extend our idea of principal square root to all other roots by saying: **the principal *n*th root** *of a number, denoted by* $\sqrt[n]{}$, *has the same sign as the number itself.*

■ Example 9–1 F

Find the indicated root.

1. $\sqrt[4]{16} = 2$, since $2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$.
2. $\sqrt[3]{-27} = -3$, since $(-3)(-3)(-3) = (-3)^3 = -27$.
3. $\sqrt[3]{-125} = -5$, since $(-5)(-5)(-5) = (-5)^3 = -125$.
4. $\sqrt{-16}$ Does not exist in the set of real numbers.

▶ ***Quick check*** Find the cube root. $\sqrt[3]{-64}$ ■

Mastery points

Can you

- Find the principal square root of a perfect-square integer?
- Find the principal root of a number?
- Find the decimal approximation with a calculator for a root that is an irrational number?

Exercise 9–1

Find the indicated square root. See examples 9–1 A and B.

Example $-\sqrt{16}$

Solution $= -4$ Since $4 \cdot 4 = 16$, and we want the negative value of the square root

1. $\sqrt{100}$	2. $\sqrt{36}$	3. $\sqrt{4}$	4. $\sqrt{64}$	5. $-\sqrt{144}$	6. $-\sqrt{81}$
7. $-\sqrt{121}$	8. $-\sqrt{4}$	9. $\sqrt{121}$	10. $\sqrt{81}$	11. $-\sqrt{16}$	12. $-\sqrt{64}$

Find the decimal approximation to three decimal places of the indicated square root by using a calculator. See example 9–1 C.

Examples $\sqrt{48}$ $-\sqrt{56}$

Solutions ≈ 6.928 ≈ -7.483

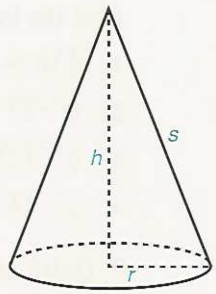
13. $\sqrt{18}$	14. $\sqrt{24}$	15. $\sqrt{41}$	16. $\sqrt{47}$	17. $-\sqrt{52}$	18. $-\sqrt{10}$

Solve the following problems. See example 9–1 D.

19. The current I (amperes) in a circuit is found by the formula $I = \sqrt{\dfrac{\text{watts}}{\text{ohms}}}$. What is the current of a circuit that has 3 ohms resistance and uses 1,728 watts?

20. What is the current of a circuit that has 2 ohms resistance and uses 450 watts? (Refer to exercise 19.)

21. The slant height, S, of a right circular cone is found by the formula $S = \sqrt{r^2 + h^2}$, where r is the radius of the base and h is the height of the cone. What is the slant height of a right circular cone whose base radius is 5 units and whose height is 12 units?



22. What is the slant height of a right circular cone whose base radius is 3 units and whose height is 4 units? (Refer to exercise 21.)

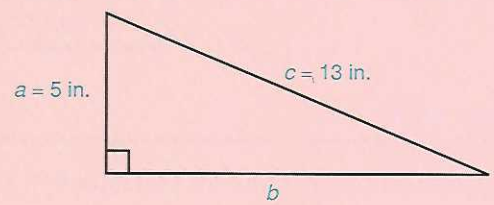
In the following right triangles, find the length of the unknown side. See example 9–1 E.

Example Find the second leg of a right triangle whose hypotenuse has length 13 inches and whose first leg is 5 inches long.

Solution We want to find b given that $c = 13$ in. and $a = 5$ in. Using one of the forms of the theorem,

$$\begin{aligned} b &= \sqrt{c^2 - a^2} \\ &= \sqrt{(13)^2 - (5)^2} && \text{Substitute} \\ &= \sqrt{169 - 25} && \text{Square the values} \\ &= \sqrt{144} = 12 \end{aligned}$$

Hence $b = 12$ in.



23. $a = 3$ m, $b = 4$ m
24. $a = 8$ ft, $c = 10$ ft
25. $a = 12$ in., $b = 5$ in.
26. $a = 15$ cm, $b = 8$ cm
27. $a = 6$ yd, $c = 10$ yd
28. $b = 16$ m, $c = 20$ m
29. $a = 12$ mm, $b = 16$ mm
30. $a = 10$ in., $b = 24$ in.

Find the indicated root. See example 9–1 F.

Example $\sqrt[3]{-64}$

Solution $= -4$ Since $(-4)^3 = -64$

31. $\sqrt[3]{8}$
32. $\sqrt[3]{27}$
33. $\sqrt[3]{125}$
34. $\sqrt[3]{64}$
35. $\sqrt[3]{-8}$
36. $\sqrt[3]{-64}$
37. $\sqrt[4]{81}$
38. $\sqrt[4]{16}$
39. $-\sqrt[4]{81}$
40. $-\sqrt[4]{625}$
41. $\sqrt[5]{32}$
42. $\sqrt[6]{64}$
43. $-\sqrt[5]{243}$
44. $\sqrt[5]{-243}$
45. $\sqrt[10]{1}$
46. $\sqrt[14]{1}$
47. $\sqrt[9]{-1}$
48. $\sqrt[15]{-1}$

Solve the following problems. See example 9–1 D.

49. The formula for finding the length of an edge, e, of a cube when the volume, v, is known is $e = \sqrt[3]{v}$. What is the length of the edge of a cube whose volume is 729 cubic units?

50. What is the length of the edge of a cube whose volume is 216 cubic units? (Refer to exercise 49.)

Review exercises

Rewrite the following numbers in prime factor form. See section 1–1.

1. 9
2. 12
3. 8
4. 40
5. 50
6. 81
7. 64
8. 16

9–2 ■ Product property for radicals

Multiplying square roots

In this section, we are going to develop properties for simplifying radicals. Consider the example

$$\sqrt{4} \cdot \sqrt{25} = 2 \cdot 5 = 10$$

We also observe that

$$\sqrt{4 \cdot 25} = \sqrt{100} = 10$$

From our example, we can conclude that

$$\sqrt{4} \cdot \sqrt{25} = \sqrt{4 \cdot 25}$$

We now can generalize the product property for square roots.

Product property for square roots

For all nonnegative real numbers a and b,

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

Concept

The product of two square roots is equal to the square root of their product.

■ **Example 9–2 A**

Perform the indicated operations. Assume that all variables represent nonnegative real numbers.

1. $\sqrt{3}\sqrt{5} = \sqrt{3 \cdot 5} = \sqrt{15}$
2. $\sqrt{6}\sqrt{7} = \sqrt{6 \cdot 7} = \sqrt{42}$
3. $\sqrt{3}\sqrt{a} = \sqrt{3a}$
4. $\sqrt{x}\sqrt{y}\sqrt{z} = \sqrt{xyz}$ ■

Simplifying principal square roots

An important use of our product property is in simplifying radicals. Consider the following example:

Since 12 can be factored into $4 \cdot 3$, by our product property, we can write

$$\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$$

We are able to simplify the radical because the radicand contains a perfect-square integer factor, 4. In our example, $2\sqrt{3}$ is called the *simplified form* of $\sqrt{12}$.

Simplifying the principal square root

1. If the radicand is a perfect-square integer, write the corresponding square root.
2. If possible, factor the radicand so that at least one factor is a perfect-square integer. Write the corresponding square root as a coefficient of the radical.
3. The square root is in simplest form when the radicand has no perfect-square integer factors other than 1.

The following property is used when changing radicals involving variables in the radicand to simplest form.

$\sqrt{a^2}$ property

If a is any nonnegative real number, then

$$\sqrt{a^2} = a$$

Example 9–2 B

Simplify the following expressions. Assume that all variables represent nonnegative real numbers.

1. $\sqrt{50} = \sqrt{25 \cdot 2}$ Factor having a perfect-square integer
$= \sqrt{25}\sqrt{2}$ Product property
$= 5\sqrt{2}$ $\sqrt{25} = 5$
2. $\sqrt{28} = \sqrt{4 \cdot 7} = \sqrt{4}\sqrt{7} = 2\sqrt{7}$
3. $\sqrt{9a} = \sqrt{9 \cdot a} = \sqrt{9}\sqrt{a} = 3\sqrt{a}$
4. $\sqrt{a^3} = \sqrt{a^2 \cdot a} = \sqrt{a^2}\sqrt{a} = a\sqrt{a}$
5. $\sqrt{a^3b^4} = \sqrt{a^2 \cdot a \cdot b^2 \cdot b^2} = \sqrt{a^2}\sqrt{a}\sqrt{b^2}\sqrt{b^2} = a\sqrt{a}bb = ab^2\sqrt{a}$
6. $\sqrt{a^2 + b^2}$ will not simplify because we are not able to factor the radicand. *The radicand must always be in a factored form before we can simplify.*

Note $\sqrt{a^2 + b^2} \neq \sqrt{a^2} + \sqrt{b^2}$. For example, $\sqrt{3^2 + 4^2} = \sqrt{9 + 16} \neq \sqrt{9} + \sqrt{16}$. Since $\sqrt{9 + 16} = \sqrt{25} = 5$, whereas $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$.

7. $\sqrt{2}\,\sqrt{6} = \sqrt{2 \cdot 6} = \sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$
8. $\sqrt{10}\sqrt{5} = \sqrt{10 \cdot 5} = \sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$
9. $\sqrt{14x}\sqrt{2x} = \sqrt{14x \cdot 2x} = \sqrt{28x^2} = \sqrt{4 \cdot 7 \cdot x^2} = \sqrt{4}\sqrt{7}\sqrt{x^2}$
$= 2\sqrt{7}x = 2x\sqrt{7}$

▶ ***Quick check*** Simplify. $\sqrt{8a^4}$ and $\sqrt{3a}\sqrt{6a}$

Multiplying nth roots

In section 9–1, we observed that even roots of negative numbers do not exist in the set of real numbers. Therefore, **we will consider all variables to be representing nonnegative real numbers whenever the index of the radical is even.**

Our product property for square roots can be extended to radicals with any index.

Product property for radicals

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$$

Concept

When we multiply two radicals *having the same index,* we multiply the radicands and put the product under a radical symbol with the common index.

Example 9–2 C

Perform the indicated operations.

1. $\sqrt[3]{3}\sqrt[3]{2} = \sqrt[3]{3 \cdot 2} = \sqrt[3]{6}$
2. $\sqrt[5]{7}\sqrt[5]{9} = \sqrt[5]{7 \cdot 9} = \sqrt[5]{63}$
3. $\sqrt[3]{3}\sqrt[4]{5}$ These radicals cannot be multiplied together in this form since they do not have the same index.

Simplifying principal nth roots

We simplify *n*th roots, where n is greater than 2, as we did square roots. As long as the radicand can be factored so that one or more factors is a

1. perfect cube when the index is 3,
2. perfect fourth-power when the index is 4,
3. perfect fifth-power when the index is 5, and so on,

the radical can be simplified. To do this, we use the property

$$\sqrt[n]{a^n} = a$$

where a is a nonnegative real number.

Example 9–2 D

Simplify the following radical expressions. Assume that all variables represent nonnegative real numbers.

1. $\sqrt[3]{81} = \sqrt[3]{27 \cdot 3}$ Factor having a perfect-cube integer
$= \sqrt[3]{27}\sqrt[3]{3}$ Product property
$= 3\sqrt[3]{3}$ $\sqrt[3]{27} = 3$
2. $\sqrt[4]{32} = \sqrt[4]{16 \cdot 2} = \sqrt[4]{16}\,\sqrt[4]{2} = 2\sqrt[4]{2}$
3. $\sqrt[3]{x^5} = \sqrt[3]{x^3 \cdot x^2} = \sqrt[3]{x^3}\,\sqrt[3]{x^2} = x\sqrt[3]{x^2}$
4. $\sqrt[5]{y^{10}} = \sqrt[5]{y^5 \cdot y^5} = \sqrt[5]{y^5}\,\sqrt[5]{y^5} = y \cdot y = y^2$

Note In example 4, the exponent 10 is evenly divisible by the index 5, and the radical is eliminated. When the exponent of a factor is evenly divisible by the index, that factor will no longer remain under the radical symbol.

5. $\sqrt[4]{x^7y^4} = \sqrt[4]{x^4 \cdot x^3 \cdot y^4} = \sqrt[4]{x^4}\,\sqrt[4]{x^3}\,\sqrt[4]{y^4} = x\sqrt[4]{x^3}y = xy\sqrt[4]{x^3}$
6. $\sqrt[3]{a^7b^2} = \sqrt[3]{a^3a^3ab^2} = \sqrt[3]{a^3}\,\sqrt[3]{a^3}\,\sqrt[3]{ab^2} = a \cdot a\sqrt[3]{ab^2} = a^2\sqrt[3]{ab^2}$

Note No simplification relative to b is possible because the exponent of b is less than the value of the index.

7. $\sqrt[3]{54x^3y^5} = \sqrt[3]{27 \cdot 2 \cdot x^3 \cdot y^3 \cdot y^2} = \sqrt[3]{27}\,\sqrt[3]{x^3}\,\sqrt[3]{y^3}\,\sqrt[3]{2y^2} = 3xy\sqrt[3]{2y^2}$

▶ ***Quick check*** Simplify. $\sqrt[3]{8a^4}$

Observe from the preceding examples that **we can simplify a radical if the radicand has a factor(s) whose exponent is equal to or greater than the index.**

■ Example 9–2 E

Perform the indicated multiplication and simplify. Assume that all variables represent nonnegative real numbers.

1. $\sqrt[3]{a^2}\,\sqrt[3]{a} = \sqrt[3]{a^2 \cdot a}$ Product property
$= \sqrt[3]{a^3}$ Multiply like bases
$= a$ Perfect cube

2. $\sqrt[4]{8a^2}\,\sqrt[4]{4a^3} = \sqrt[4]{8a^2 \cdot 4a^3} = \sqrt[4]{32a^5} = \sqrt[4]{16 \cdot 2 \cdot a^4 \cdot a}$
$= \sqrt[4]{16}\,\sqrt[4]{a^4}\,\sqrt[4]{2a} = 2a\sqrt[4]{2a}$

▶ ***Quick check*** Simplify. $\sqrt[3]{9a^2}\sqrt[3]{3a^2}$ ■

Note A very common error in problems involving radicals is to forget to carry along the correct index for the radical symbol.

Mastery points

Can you

- Multiply radicals having the same index?
- Simplify radicals?

Exercise 9–2

Perform any indicated operations and simplify. Assume that all variables represent nonnegative real numbers. See examples 9–2 A and B.

Examples $\sqrt{8a^4}$

Solutions
$= \sqrt{4 \cdot 2a^2a^2}$ Factor having perfect squares
$= \sqrt{4}\sqrt{a^2}\sqrt{a^2}\sqrt{2}$ Product property
$= 2aa\sqrt{2}$ $\sqrt{4} = 2,\ \sqrt{a^2} = a$
$= 2a^2\sqrt{2}$ Multiply

$\sqrt{3a}\sqrt{6a}$
$= \sqrt{3a \cdot 6a}$ Product property
$= \sqrt{18a^2}$ Multiply
$= \sqrt{9 \cdot 2 \cdot a^2}$ Factor having perfect squares
$= \sqrt{9}\sqrt{2}\sqrt{a^2}$ Product property
$= 3\sqrt{2}\,a$ $\sqrt{9} = 3,\ \sqrt{a^2} = a$
$= 3a\sqrt{2}$ Multiply

1. $\sqrt{16}$
2. $\sqrt{63}$
3. $\sqrt{20}$
4. $\sqrt{75}$
5. $\sqrt{45}$
6. $\sqrt{48}$
7. $\sqrt{32}$
8. $\sqrt{27}$
9. $\sqrt{80}$
10. $\sqrt{54}$
11. $\sqrt{98}$
12. $\sqrt{96}$
13. $\sqrt{a^7}$
14. $\sqrt{a^5}$
15. $\sqrt{4a^2b^3}$
16. $\sqrt{9ab^4c^3}$
17. $\sqrt{27a^3b^5}$
18. $\sqrt{24x^5yz^3}$
19. $\sqrt{6}\sqrt{3}$
20. $\sqrt{27}\sqrt{6}$
21. $\sqrt{15}\sqrt{15}$
22. $\sqrt{11}\sqrt{11}$
23. $\sqrt{6}\sqrt{10}$
24. $\sqrt{18}\sqrt{24}$
25. $\sqrt{25}\sqrt{15}$
26. $\sqrt{20}\sqrt{20}$
27. $\sqrt{5}\sqrt{15}$
28. $\sqrt{2a}\sqrt{3a}$
29. $\sqrt{5x}\sqrt{15x}$
30. $\sqrt{6x}\sqrt{14xy}$
31. $\sqrt{2a}\sqrt{24b^2}$
32. A square-shaped television picture tube has a surface area of 121 square inches. What is the length of the side of the tube? (*Hint:* Area of a square is found by squaring the length of a side. $A = s^2$.)
33. A room in the shape of a square is 169 square feet. What is the length of a side? (See exercise 32.)
34. The formula for approximating the velocity V in miles per hour of a car based on the length of its skid marks S (in feet) on wet pavement is given by $V = 2\sqrt{3S}$. If the skid marks are 75 feet long, what was the velocity?
35. The formula for approximating the velocity V in miles per hour of a car based on the length of its skid marks S (in feet) on dry pavement is given by $V = 2\sqrt{6S}$. If the skid marks are 24 feet long, what was the velocity?

Perform any indicated operations and simplify. Assume that all variables represent positive real numbers. See examples 9–2 D and E.

Examples $\sqrt[3]{8a^4}$

Solutions
$= \sqrt[3]{8 \cdot a^3 \cdot a}$ Factor having perfect cubes
$= \sqrt[3]{8}\,\sqrt[3]{a^3}\,\sqrt[3]{a}$ Product property
$= 2a\sqrt[3]{a}$ $\sqrt[3]{8} = 2$, $\sqrt[3]{a^3} = a$

$\sqrt[3]{9a^2}\,\sqrt[3]{3a^2}$
$= \sqrt[3]{9a^2 \cdot 3a^2}$ Product property
$= \sqrt[3]{27a^4}$ Multiply radicands
$= \sqrt[3]{27 \cdot a^3 \cdot a}$ Factor having perfect cubes
$= \sqrt[3]{27}\,\sqrt[3]{a^3}\,\sqrt[3]{a}$ Product property
$= 3a\sqrt[3]{a}$ $\sqrt[3]{27} = 3$, $\sqrt[3]{a^3} = a$

36. $\sqrt[3]{48}$ **37.** $\sqrt[5]{64}$ **38.** $\sqrt[4]{32}$ **39.** $\sqrt[3]{24}$ **40.** $\sqrt[5]{a^7}$
41. $\sqrt[3]{b^8}$ $\boxed{42.}$ $\sqrt[4]{x^9}$ **43.** $\sqrt[5]{y^{15}}$ **44.** $\sqrt[3]{a^{12}}$ **45.** $\sqrt[3]{4a^2b^3}$
46. $\sqrt[3]{8r^2s^8}$ **47.** $\sqrt[3]{16a^4b^5}$ **48.** $\sqrt[5]{64x^{10}y^{14}}$ **49.** $\sqrt[3]{81a^5b^{11}}$ **50.** $\sqrt[3]{a^2}\,\sqrt[3]{a}$
51. $\sqrt[3]{b^2}\,\sqrt[3]{b^2}$ **52.** $\sqrt[5]{b^4}\,\sqrt[5]{b^3}$ **53.** $\sqrt[5]{a}\,\sqrt[5]{a^4}$ $\boxed{54.}$ $\sqrt[3]{5a^2b}\,\sqrt[3]{75a^2b^2}$
55. $\sqrt[3]{3ab^2}\,\sqrt[3]{18a^2b^2}$ **56.** $\sqrt[4]{8a^3b}\,\sqrt[4]{4a^2b^2}$ **57.** $\sqrt[4]{27a^2b^3}\,\sqrt[4]{9ab}$ **58.** $\sqrt[3]{25x^5y^7}\,\sqrt[3]{15xy^3}$
59. $\sqrt[3]{16a^{11}b^4}\,\sqrt[3]{12a^4b^6}$ **60.** $\sqrt[4]{8xy}\,\sqrt[4]{4x^3y^3}$

$\boxed{61.}$ The moment of inertia for a rectangle is given by the formula $I = \dfrac{bh^3}{12}$. If we know the values of I and b, we can solve for h as follows: $h = \sqrt[3]{\dfrac{12I}{b}}$. Find h if $I = 2$ in.^{4} and $b = 3$ in.

62. Use exercise 61 to find h if $I = 27$ in.^{4} and $b = 4$ in.

63. The moment of inertia for a circle is given by the formula $I = \dfrac{\pi r^4}{4}$. If we know the value of I, we can solve for r as follows: $r = \sqrt[4]{\dfrac{4I}{\pi}}$. Find r if $I = 12.56$ in.^{4} and we use 3.14 for π.

64. Use exercise 63 to find r if $I = 63.585$ in.^{4}

Review exercises

Reduce the following fractions and rational expressions to lowest terms. Assume that no variable is equal to zero. See sections 1–1, 3–3, and 5–2.

1. $\dfrac{49}{56}$ **2.** $\dfrac{16x^3y^2}{-4xy}$ **3.** $\dfrac{2y^2 - 50}{y^2 - 4y - 5}$

Perform the indicated operations and simplify. Assume that all variables represent nonnegative real numbers. See sections 9–1 and 9–2.

4. $\sqrt{5}\sqrt{5}$ **5.** $\sqrt{3}\sqrt{3}$ **6.** $\sqrt{2}\sqrt{8}$ **7.** $\sqrt{12}\sqrt{3}$ **8.** $\sqrt{x}\sqrt{x}$

9–3 ■ Quotient property for radicals

The square root of a fraction

The following example will help us develop a property for division involving radicals.

$$\sqrt{\frac{4}{9}} = \sqrt{\left(\frac{2}{3}\right)^2} = \frac{2}{3}$$

We also observe that

$$\frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$$

From our example, we can conclude that

$$\sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}}$$

We can now generalize this idea.

Quotient property for square roots

For any nonnegative real numbers a and b, where $b \neq 0$,

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Concept

The square root of a fraction can be written as the square root of the numerator divided by the square root of the denominator.

Example 9–3 A

Simplify the following expressions. Assume that all variables represent positive real numbers.

1. $\sqrt{\frac{16}{25}} = \frac{\sqrt{16}}{\sqrt{25}}$ Rewrite as the square root of the numerator over the square root of the denominator and simplify
$= \frac{4}{5}$

2. $\sqrt{\frac{36}{49}} = \frac{\sqrt{36}}{\sqrt{49}} = \frac{6}{7}$

3. $\sqrt{\frac{81}{100}} = \frac{\sqrt{81}}{\sqrt{100}} = \frac{9}{10}$

4. $\sqrt{\frac{x^4}{64}} = \frac{\sqrt{x^4}}{\sqrt{64}} = \frac{x^2}{8}$

5. $\sqrt{\frac{x^3}{y^4}} = \frac{\sqrt{x^3}}{\sqrt{y^4}} = \frac{x\sqrt{x}}{y^2}$

▶ ***Quick check*** Simplify. $\sqrt{\frac{16}{49}}$ and $\sqrt{\frac{a^2}{b}}$ ■

Rationalizing the denominator

When simplifying and evaluating radical expressions containing a radical in the denominator, it is easier if we can eliminate the radical in the denominator. For example,

$$\sqrt{\frac{4}{5}} = \frac{\sqrt{4}}{\sqrt{5}} = \frac{2}{\sqrt{5}}$$

Since $\sqrt{5} \cdot \sqrt{5} = 5$, we can eliminate the radical $\sqrt{5}$ in the denominator by multiplying the numerator and the denominator of the fraction by $\sqrt{5}$.

$$\frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{\sqrt{25}} = \frac{2\sqrt{5}}{5}$$

The process of changing the denominator from a radical to a rational number is called **rationalizing the denominator.**

Rationalizing the denominator

1. Multiply the numerator and the denominator by the square root that is in the denominator. The radicand in the denominator will be a perfect-square integer.
2. Simplify the radical expressions in the numerator and the denominator.
3. Reduce the resulting fraction if possible.

Example 9–3 B

Simplify the following expressions. Leave no radicals in the denominator. Assume that all variables represent positive real numbers.

1. $\dfrac{5}{\sqrt{7}} = \dfrac{5}{\sqrt{7}} \cdot \dfrac{\sqrt{7}}{\sqrt{7}}$ Multiply numerator and denominator by $\sqrt{7}$

$= \dfrac{5\sqrt{7}}{\sqrt{49}}$ Multiply in numerator and denominator

$= \dfrac{5\sqrt{7}}{7}$ $\sqrt{49} = 7$

2. $\dfrac{4}{\sqrt{6}} = \dfrac{4}{\sqrt{6}} \cdot \dfrac{\sqrt{6}}{\sqrt{6}} = \dfrac{4\sqrt{6}}{\sqrt{36}} = \dfrac{4\sqrt{6}}{6} = \dfrac{2\sqrt{6}}{3}$

Note In example 2, we were able to reduce the fraction as a final step. Always check to see that the answer is in reduced form.

3. $\dfrac{a}{\sqrt{a}} = \dfrac{a}{\sqrt{a}} \cdot \dfrac{\sqrt{a}}{\sqrt{a}} = \dfrac{a\sqrt{a}}{\sqrt{a^2}} = \dfrac{a\sqrt{a}}{a} = \sqrt{a}$

4. $\sqrt{\dfrac{a^3}{b}} = \dfrac{\sqrt{a^3}}{\sqrt{b}} = \dfrac{a\sqrt{a}}{\sqrt{b}} \cdot \dfrac{\sqrt{b}}{\sqrt{b}} = \dfrac{a\sqrt{ab}}{\sqrt{b^2}} = \dfrac{a\sqrt{ab}}{b}$

▶ ***Quick check*** Simplify. $\dfrac{6}{\sqrt{6}}$

The following is a summary of the conditions necessary for a radical expression to be in **simplest form,** also called **standard form.**

1. The radicand contains no factors that can be written with an exponent greater than or equal to the index. ($\sqrt{a^3}$ violates this.)
2. The radicand contains no fractions. $\left(\sqrt{\dfrac{a}{b}}\text{ violates this.}\right)$
3. No radicals appear in the denominator. $\left(\dfrac{1}{\sqrt{a}}\text{ violates this.}\right)$

The nth root of a fraction

Our quotient property for square roots can be extended to radicals with any index.

Quotient property for radicals

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \qquad (b \neq 0)$$

Concept

The nth root of a fraction can be written as the nth root of the numerator divided by the nth root of the denominator.

Example 9–3 C

Simplify the following expressions. Assume that all variables represent positive real numbers.

1. $\sqrt[3]{\frac{8}{27}} = \frac{\sqrt[3]{8}}{\sqrt[3]{27}}$ Rewrite as the cube root of the numerator over the cube root of the denominator and simplify

$= \frac{2}{3}$

2. $\sqrt[5]{\frac{32}{a^5}} = \frac{\sqrt[5]{32}}{\sqrt[5]{a^5}} = \frac{2}{a}$

3. $\sqrt[3]{\frac{x^5}{y^6}} = \frac{\sqrt[3]{x^5}}{\sqrt[3]{y^6}} = \frac{x\sqrt[3]{x^2}}{y^2}$

▶ ***Quick check*** Simplify. $\sqrt[3]{\frac{1}{27}}$ and $\sqrt[3]{\frac{8a^5}{b^3}}$ ■

Rationalizing the denominator (nth root)

The following example will help us develop a general rule for rationalizing a denominator that has a single term.

$$\sqrt[3]{\frac{1}{a}} = \frac{\sqrt[3]{1}}{\sqrt[3]{a}} = \frac{1}{\sqrt[3]{a}}$$

At this point, a radical still remains in the denominator. We must now determine what we can do to the fraction to remove the radical from the denominator.

Observations:

1. We can multiply the numerator and the denominator by the same number and form equivalent fractions.
2. If we multiply by a radical, the indices must be the same to carry out the multiplication.
3. To bring a factor out from under the radical symbol and not leave any of the factor behind, the index must divide evenly into the exponent.

With these observations in mind, we rationalize the fraction as follows:

$= \frac{1}{\sqrt[3]{a}} \cdot \frac{\sqrt[3]{}}{\sqrt[3]{}}$ Indices are the same

$= \frac{1}{\sqrt[3]{a}} \cdot \frac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}}$ Multiply numerator and denominator by the same number

$$= \frac{\sqrt[3]{a^2}}{\sqrt[3]{a^3}}$$ The sum of the exponents of a in the denominator is equal to the index

$$= \frac{\sqrt[3]{a^2}}{a}$$ The index divides evenly into the exponent, the radical is eliminated

To rationalize an nth root denominator

1. Multiply the numerator and the denominator by a radical with the same index as the radical that we wish to eliminate from the denominator.
2. The exponent of each factor under the radical must be such that when we add it to the original exponent of the factor under the radical in the denominator, the sum will be equal to or divisible by the index of the radical.
3. Carry out the multiplication and reduce the fraction if possible.

Example 9–3 D

Simplify the following expressions. Leave no radicals in the denominator. Assume that all variables represent positive real numbers.

1. $\frac{1}{\sqrt[3]{7}} = \frac{1}{\sqrt[3]{7}} \cdot \frac{\sqrt[3]{7^2}}{\sqrt[3]{7^2}}$ Multiply numerator and denominator by $\sqrt[3]{7^2}$

$= \frac{\sqrt[3]{7^2}}{\sqrt[3]{7^3}}$ Multiply in numerator and denominator ($\sqrt[3]{7}\sqrt[3]{7^2} = \sqrt[3]{7^3}$)

$= \frac{\sqrt[3]{7^2}}{7}$ $\sqrt[3]{7^3} = 7$

$= \frac{\sqrt[3]{49}}{7}$ $7^2 = 49$

2. $\frac{a}{\sqrt[5]{b^2}} = \frac{a}{\sqrt[5]{b^2}} \cdot \frac{\sqrt[5]{b^3}}{\sqrt[5]{b^3}} = \frac{a\sqrt[5]{b^3}}{\sqrt[5]{b^5}} = \frac{a\sqrt[5]{b^3}}{b}$

3. $\frac{x}{\sqrt[4]{x}} = \frac{x}{\sqrt[4]{x}} \cdot \frac{\sqrt[4]{x^3}}{\sqrt[4]{x^3}} = \frac{x\sqrt[4]{x^3}}{\sqrt[4]{x^4}} = \frac{x\sqrt[4]{x^3}}{x} = \sqrt[4]{x^3}$

4. $\frac{1}{\sqrt[5]{a^2b}} = \frac{1}{\sqrt[5]{a^2b}} \cdot \frac{\sqrt[5]{a^3b^4}}{\sqrt[5]{a^3b^4}} = \frac{\sqrt[5]{a^3b^4}}{\sqrt[5]{a^5b^5}} = \frac{\sqrt[5]{a^3b^4}}{ab}$

▶ ***Quick check*** Simplify. $\frac{1}{\sqrt[5]{b^2}}$

Mastery points

Can you

- Simplify radicals containing fractions?
- Rationalize denominators?

Exercise 9–3

Simplify the following expressions. Leave no radicals in the denominator. Assume that all variables represent positive real numbers. See examples 9–3 A and B.

Examples $\sqrt{\frac{16}{49}}$ $\qquad$ $\sqrt{\frac{a^2}{b}}$ $\qquad$ $\frac{6}{\sqrt{6}}$

Solutions

$= \frac{\sqrt{16}}{\sqrt{49}}$ $= \frac{4}{7}$

Rewrite as the square root of the numerator over the square root of the denominator and simplify

$= \frac{\sqrt{a^2}}{\sqrt{b}} = \frac{a}{\sqrt{b}}\cdot\frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$

$= \frac{6}{\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}} = \frac{6\sqrt{6}}{6} = \sqrt{6}$

Multiply the numerator and the denominator by the square root in the denominator and simplify

1. $\sqrt{\frac{9}{25}}$ 2. $\sqrt{\frac{25}{36}}$ 3. $\sqrt{\frac{25}{49}}$ 4. $\sqrt{\frac{81}{100}}$ 5. $\sqrt{\frac{3}{4}}$

6. $\sqrt{\frac{5}{9}}$ 7. $\sqrt{\frac{64}{a^2}}$ 8. $\sqrt{\frac{y^4}{16}}$ 9. $\sqrt{\frac{1}{2}}$ 10. $\sqrt{\frac{1}{3}}$

11. $\sqrt{\frac{4}{7}}$ 12. $\sqrt{\frac{9}{11}}$ 13. $\sqrt{\frac{1}{15}}$ 14. $\sqrt{\frac{1}{14}}$ $\boxed{15.}$ $\sqrt{\frac{4}{75}}$

16. $\sqrt{\frac{5}{12}}$ 17. $\frac{2}{\sqrt{2}}$ 18. $\frac{6}{\sqrt{3}}$ 19. $\frac{10}{\sqrt{8}}$ 20. $\frac{15}{\sqrt{27}}$

21. $\sqrt{\frac{x^2}{y}}$ 22. $\sqrt{\frac{1}{a}}$ 23. $\sqrt{\frac{1}{x}}$ 24. $\sqrt{\frac{a^2}{b^3}}$ 25. $\frac{\sqrt{a^5}}{\sqrt{a}}$

26. Find the width of a rectangle whose diagonal is 17 feet and length is 8 feet.

27. Find the diagonal of a rectangle whose length is 5 meters and whose width is 4 meters.

28. A 13-foot ladder is placed against the wall of a house. If the bottom of the ladder is 5 feet from the house, how far from the ground is the top of the ladder?

29. At an altitude of h feet above the sea or level ground, the distance d in miles that a person can see an object is found by using the equation $d = \sqrt{\frac{3h}{2}}$. How far can someone see who is in a tower 96 feet above the ground?

Simplify the following expressions. Leave no radicals in the denominator. Assume that all variables represent positive real numbers. See examples 9–3 C and D.

Examples $\sqrt[3]{\frac{1}{27}}$ $\qquad$ $\sqrt[3]{\frac{8a^5}{b^3}}$ $\qquad$ $\frac{1}{\sqrt[5]{b^2}}$

Solutions

$= \frac{\sqrt[3]{1}}{\sqrt[3]{27}} = \frac{1}{3}$

$= \frac{\sqrt[3]{8a^5}}{\sqrt[3]{b^3}} = \frac{\sqrt[3]{2^3a^5}}{b} = \frac{2a\sqrt[3]{a^2}}{b}$

Rewrite as the cube root of the numerator over the cube root of the denominator and simplify

$= \frac{1}{\sqrt[5]{b^2}}\cdot\frac{\sqrt[5]{b^3}}{\sqrt[5]{b^3}}$ Multiply the numerator and the denominator by $\sqrt[5]{b^3}$

$= \frac{\sqrt[5]{b^3}}{\sqrt[5]{b^5}}$ Perfect 5th root

$= \frac{\sqrt[5]{b^3}}{b}$ $\sqrt[5]{b^5} = b$

30. $\sqrt[3]{\frac{8}{27}}$ 31. $\sqrt[3]{\frac{1}{8}}$ 32. $\sqrt[4]{\frac{16}{81}}$ 33. $\sqrt[3]{\frac{27}{125}}$ 34. $\sqrt[3]{\frac{a^2}{b^2}}$

35. $\sqrt[3]{\frac{3a^6}{b^3}}$ 36. $\sqrt[3]{\frac{x}{y^{12}}}$ 37. $\sqrt[5]{\frac{a^4}{b^{10}}}$ 38. $\sqrt[5]{\frac{32x^4}{y^5}}$ 39. $\sqrt[4]{\frac{a^4b^9}{c^{11}}}$

40. $\sqrt[4]{\frac{a^9b^{13}}{c^8}}$ 41. $\sqrt[5]{\frac{x^3y^2}{z^{15}}}$ 42. $\sqrt[3]{\frac{8}{9}}$ 43. $\sqrt[3]{\frac{4}{25}}$ 44. $\sqrt[3]{\frac{27}{16}}$

45. $\sqrt[4]{\frac{16}{125}}$ 46. $\sqrt[4]{\frac{3}{4}}$ 47. $\sqrt[3]{\frac{x^3}{y^2}}$ 48. $\sqrt[3]{\frac{x^6}{y}}$ 49. $\frac{ab}{\sqrt[3]{a^2}}$

50. $\frac{xy}{\sqrt[5]{y^3}}$ 51. $\sqrt[3]{\frac{a^3}{b^2c}}$ 52. $\sqrt[3]{\frac{8}{xy^2}}$ 53. $\sqrt[3]{\frac{a^2}{b^2c}}$ 54. $\frac{a}{\sqrt[5]{a^2b^4}}$

55. $\frac{ab}{\sqrt[3]{ab^2}}$ 56. $\frac{xy^2}{\sqrt[5]{x^4y}}$

57. If we wish to construct a sphere of specific volume, we can find the length of radius necessary by the formula $r = \sqrt[3]{\frac{3V}{4\pi}}$. Find the radius necessary for a sphere to have a volume of 113.04 cubic units. (Use 3.14 for π.)

58. Use exercise 57 to find r if $V = 904.32$ cubic units. (Use 3.14 for π.)

Review exercises

Combine in the following. See section 2–3.

1. $4x + 2x$ 2. $9y - 5y$ 3. $5ab + 3ab$ 4. $xy + 4xy$

Multiply the following. See section 3–2.

5. $(x + 3)(x - 3)$ 6. $(x + y)(x - y)$

9–4 ■ Sums and differences of radicals

Like radicals

We have learned that in addition and subtraction of algebraic expressions, we can only combine like terms. This same idea applies when we are dealing with radicals. **We can add or subtract only like radicals.** Like radicals are radicals having the same index and the same radicand. For example, $3\sqrt{5x}$ and $-2\sqrt{5x}$ are like radicals, but $5\sqrt{7x}$ and $7\sqrt{5x}$ are not, because the radicands are not the same.

Addition and subtraction involving square roots

Addition and subtraction of radicals follow the same procedure as addition and subtraction of algebraic expressions. That is, *once we have determined that we have like radicals, the operations of addition and subtraction are performed only with the numerical coefficients.*

Example 9–4 A

Perform the indicated operations and simplify. Assume that all variables represent nonnegative real numbers.

1. $5\sqrt{2} + 3\sqrt{2} = (5 + 3)\sqrt{2} = 8\sqrt{2}$ — Apply the distributive property
2. $12\sqrt{3} - \sqrt{3} = (12 - 1)\sqrt{3} = 11\sqrt{3}$
3. $2\sqrt{a} + 3\sqrt{a} = 5\sqrt{a}$

▶ ***Quick check*** Simplify. $3\sqrt{6} + 2\sqrt{6} - \sqrt{6}$ ■

Consider the example

$$\sqrt{27} + 4\sqrt{3}$$

It appears that the indicated addition cannot be performed since we do not have like radicals. However we should have observed that the $\sqrt{27}$ can be simplified as

$$\sqrt{27} = \sqrt{9 \cdot 3} = 3\sqrt{3}$$

Our problem then becomes

$$\sqrt{27} + 4\sqrt{3} = 3\sqrt{3} + 4\sqrt{3} = 7\sqrt{3}$$

and we are able to add the like radicals. Therefore, *whenever we are working with radicals, we must be certain that all radicals are in simplest form.*

To combine like radicals

1. Perform any simplification within the terms.
2. Use the distributive property to combine terms that have like radicals.

Example 9–4 B

Perform the indicated operations. Assume that all variables represent nonnegative real numbers.

1. $\sqrt{45} + \sqrt{20} = \sqrt{9 \cdot 5} + \sqrt{4 \cdot 5}$ — Factor $45 = 9 \cdot 5$ and $20 = 4 \cdot 5$

$= 3\sqrt{5} + 2\sqrt{5}$ — $\sqrt{4} = 2$; $\sqrt{9} = 3$

$= 5\sqrt{5}$ — Add coefficients

2. $\sqrt{32} + 5\sqrt{8} = \sqrt{16 \cdot 2} + 5\sqrt{4 \cdot 2}$ — Factor $32 = 16 \cdot 2$; $8 = 4 \cdot 2$

$= 4\sqrt{2} + 5 \cdot 2\sqrt{2}$ — $\sqrt{16} = 4$ and $\sqrt{4} = 2$

$= 4\sqrt{2} + 10\sqrt{2}$ — Multiply $5 \cdot 2 = 10$

$= 14\sqrt{2}$ — Add coefficients

3. $3\sqrt{3a} - \sqrt{12a} + 5\sqrt{48a}$

$= 3\sqrt{3a} - \sqrt{4 \cdot 3a} + 5\sqrt{16 \cdot 3a}$ — Factor $12 = 4 \cdot 3$; $48 = 16 \cdot 3$

$= 3\sqrt{3a} - 2\sqrt{3a} + 5 \cdot 4\sqrt{3a}$ — $\sqrt{4} = 2$; $\sqrt{16} = 4$

$= 3\sqrt{3a} - 2\sqrt{3a} + 20\sqrt{3a}$ — $5 \cdot 4 = 20$

$= 21\sqrt{3a}$ — Combine coefficients

▶ ***Quick check*** Simplify. $5\sqrt{2} + \sqrt{18}$ ■

Addition and subtraction involving nth roots

Addition and subtraction of radicals other than square roots follow the same procedure as addition and subtraction of expressions containing square roots. That is, *once we have determined that we have like radicals, the operations of addition and subtraction are performed only with the numerical coefficients.*

Example 9–4 C

Perform the indicated operations and simplify. Assume that all variables represent nonnegative real numbers.

1. $\sqrt[3]{5} + 6\sqrt[3]{5} = (1 + 6)\sqrt[3]{5} = 7\sqrt[3]{5}$ — Combine coefficients

2. $4\sqrt[3]{81} - \sqrt[3]{24} = 4\sqrt[3]{27 \cdot 3} - \sqrt[3]{8 \cdot 3}$ — Factor $81 = 27 \cdot 3$; $24 = 8 \cdot 3$
$= 4 \cdot 3\sqrt[3]{3} - 2\sqrt[3]{3}$ — $\sqrt[3]{27} = 3$; $\sqrt[3]{8} = 2$
$= 12\sqrt[3]{3} - 2\sqrt[3]{3}$ — $4 \cdot 3 = 12$
$= 10\sqrt[3]{3}$ — Subtract coefficients

3. $\sqrt[3]{16x^2y} + \sqrt[3]{54x^2y} = \sqrt[3]{8 \cdot 2x^2y} + \sqrt[3]{27 \cdot 2x^2y}$ — Factor $16 = 8 \cdot 2$; $54 = 27 \cdot 2$
$= 2\sqrt[3]{2x^2y} + 3\sqrt[3]{2x^2y}$ — $\sqrt[3]{8} = 2$; $\sqrt[3]{27} = 3$
$= 5\sqrt[3]{2x^2y}$ — Add coefficients

Mastery points

Can you

- Identify like radicals?
- Add and subtract like radicals?

Exercise 9–4

Perform the indicated operations and simplify. Assume that all variables represent nonnegative real numbers. See examples 9–4 A and B.

Examples $3\sqrt{6} + 2\sqrt{6} - \sqrt{6}$

Solutions $= (3 + 2 - 1)\sqrt{6}$ — Distributive property
$= 4\sqrt{6}$ — Combine coefficients

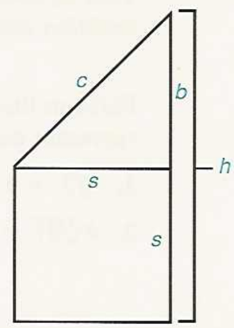
$5\sqrt{2} + \sqrt{18}$
$= 5\sqrt{2} + \sqrt{9 \cdot 2}$ — Factor $18 = 9 \cdot 2$
$= 5\sqrt{2} + 3\sqrt{2}$ — $\sqrt{9} = 3$
$= (5 + 3)\sqrt{2}$ — Distributive property
$= 8\sqrt{2}$ — Add coefficients

1. $5\sqrt{3} + 4\sqrt{3}$
2. $8\sqrt{7} - 2\sqrt{7}$
3. $6\sqrt{5} + 4\sqrt{5}$
4. $9\sqrt{6} - 6\sqrt{6}$
5. $2\sqrt{3} + 7\sqrt{3} - 3\sqrt{3}$
6. $5\sqrt{5} - 4\sqrt{5} + 6\sqrt{5}$
7. $\sqrt{7} + 5\sqrt{7} - 3\sqrt{7}$
8. $2\sqrt{10} + 11\sqrt{10} - 9\sqrt{10}$
9. $\sqrt{a} + 2\sqrt{a}$
10. $3\sqrt{x} + 4\sqrt{x}$
11. $5\sqrt{a} - 4\sqrt{a} + 7\sqrt{a}$
12. $6\sqrt{y} - \sqrt{y} + 4\sqrt{y}$
13. $5\sqrt{xy} + 2\sqrt{xy}$
14. $3\sqrt{x} + 2\sqrt{y} - \sqrt{x}$
15. $5\sqrt{a} + 2\sqrt{ab} + 3\sqrt{a}$
16. $\sqrt{ab} + 2\sqrt{ab} + 3\sqrt{a}$
17. $5\sqrt{xy} - \sqrt{xy} + 3\sqrt{y}$
18. $\sqrt{20} + 3\sqrt{5}$
19. $\sqrt{8} + 5\sqrt{2}$
20. $\sqrt{12} + \sqrt{75}$
21. $\sqrt{48} - \sqrt{27}$
22. $2\sqrt{3} + 3\sqrt{12}$
23. $5\sqrt{7} + 4\sqrt{63}$
24. $5\sqrt{3} + \sqrt{27} - \sqrt{12}$
25. $4\sqrt{2} - \sqrt{8} + \sqrt{50}$
26. $\sqrt{75} - 4\sqrt{3} + 2\sqrt{27}$
27. $\sqrt{12} + \sqrt{18} + \sqrt{50}$

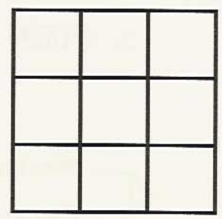
28. $\sqrt{63} - \sqrt{28} + \sqrt{24}$
29. $\sqrt{50a} + \sqrt{8a}$
30. $\sqrt{32a} - \sqrt{18a}$
31. $3\sqrt{9x} - 5\sqrt{4x}$
32. $2\sqrt{4x^2y} + 3\sqrt{25x^2y}$
33. $2\sqrt{8a} + 4\sqrt{50a} - 7\sqrt{2a}$
34. $3\sqrt{48b} - 2\sqrt{12b} + \sqrt{3b}$
35. $\sqrt{50a} + 3\sqrt{12a} - \sqrt{18a}$
36. $4\sqrt{25x^2y} + 3\sqrt{81x^2y} - 2\sqrt{2y}$

37. We can find the height, h, of the given figure by finding b from the formula $b = \sqrt{c^2 - s^2}$. If $c = 10$ units and $s = 6$ units, find h.

38. Use exercise 37 to find the height of the figure if $c = 13$ feet and $s = 5$ feet.



39. The figure is made up of 9 equal squares in which each square has an area of 7.29 square units. What are the dimensions of the figure?



Perform the indicated operations and simplify. Assume that all variables represent nonnegative real numbers. See example 9–4 C.

40. $3\sqrt[3]{4} + 5\sqrt[3]{4}$
41. $7\sqrt[5]{2} - 4\sqrt[5]{2} + 3\sqrt[5]{2}$
42. $9\sqrt[4]{3} + 6\sqrt[4]{3} + 2\sqrt[4]{3}$
43. $\sqrt[3]{16} + \sqrt[3]{54}$
44. $\sqrt[3]{24} - \sqrt[3]{81}$
45. $\sqrt[3]{81} + 2\sqrt[3]{250}$
46. $\sqrt[3]{8a^2} + \sqrt[3]{27a^2}$
47. $\sqrt[4]{16x^3} + \sqrt[4]{81x^3}$
48. $\sqrt[4]{625a} - \sqrt[4]{81a}$
49. $\sqrt[3]{64x^2y} - \sqrt[3]{27x^2y}$
50. $\sqrt[3]{x^6y} + 2x^2\sqrt[3]{y}$
51. $3a\sqrt[3]{b^2} - \sqrt[3]{a^3b^2}$
52. $\sqrt[3]{16a^2b} + \sqrt[3]{54a^2b}$

Review exercises

Multiply the following expressions. See section 3–2.

1. $3x(2x - y)$
2. $2a^2(a^2 - b^2)$
3. $(x - 1)(x - 1)$
4. $(y - 1)(y + 1)$
5. $(2x + 1)(2x - 1)$
6. $(x + 3y)(x + 2y)$
7. $(x - y)^2$
8. $(a + 2b)^2$

9–5 ■ Further operations with radicals

Multiplication of radical expressions

In section 9–2, we learned the procedure for multiplying two radicals. We now combine those ideas along with the *distributive property*, $a(b + c) = ab + ac$, to perform multiplication of radical expressions containing more than one term.

■ ***Example 9–5 A***

Perform the indicated operations and simplify. Assume that all variables represent nonnegative real numbers.

1. $\sqrt{3}(3+\sqrt{3}) = \sqrt{3}\cdot 3 + \sqrt{3}\sqrt{3}$ — Distributive property
$= 3\sqrt{3} + \sqrt{9}$ — $\sqrt{3}\sqrt{3} = \sqrt{9}$
$= 3\sqrt{3} + 3$ — $\sqrt{9} = 3$

2. $\sqrt{3}(\sqrt{6}+\sqrt{21}) = \sqrt{3}\sqrt{6} + \sqrt{3}\sqrt{21}$ — Distributive property
$= \sqrt{18} + \sqrt{63}$ — Multiply radicands
$= \sqrt{9\cdot 2} + \sqrt{9\cdot 7}$ — Factor $18 = 9\cdot 2$; $63 = 9\cdot 7$
$= 3\sqrt{2} + 3\sqrt{7}$ — Simplify radicals

3. $(\sqrt{2}+\sqrt{3})(\sqrt{2}+5\sqrt{3})$

Note In this example, we are multiplying two binomials. Therefore, as we did in chapter 3, we will *multiply each term in the first parentheses by each term in the second parentheses.*

$= \sqrt{2}\sqrt{2} + \sqrt{2}\cdot 5\sqrt{3} + \sqrt{3}\sqrt{2} + \sqrt{3}\cdot 5\sqrt{3}$ — Distributive property
$= \sqrt{4} + 5\sqrt{6} + \sqrt{6} + 5\cdot\sqrt{9}$ — Multiply radicands
$= 2 + 5\sqrt{6} + \sqrt{6} + 5\cdot 3$ — $\sqrt{4} = 2$, $\sqrt{9} = 3$
$= 2 + 5\sqrt{6} + \sqrt{6} + 15$
$= 17 + 6\sqrt{6}$ — Combine like terms

4. $(3-\sqrt{2})(3+\sqrt{2}) = 9 + 3\sqrt{2} - 3\sqrt{2} - \sqrt{4}$ — Distributive property
$= 9 + 3\sqrt{2} - 3\sqrt{2} - 2$ — Simplify radicals
$= 9 - 2$ — Combine like terms
$= 7$ — Subtract

We observe that when we simplified, there were no longer any radicals in the answer.

5. $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$
$= \sqrt{a}\sqrt{a} + \sqrt{a}\sqrt{b} - \sqrt{b}\sqrt{a} - \sqrt{b}\sqrt{b}$ — Distributive property
$= \sqrt{a^2} + \sqrt{ab} - \sqrt{ab} - \sqrt{b^2}$ — Multiply radicands
$= a + \sqrt{ab} - \sqrt{ab} - b$ — $\sqrt{a^2} = a$ and $\sqrt{b^2} = b$
$= a - b$ — Combine like terms

6. $(\sqrt{3}+2\sqrt{2})^2$
$= (\sqrt{3}+2\sqrt{2})(\sqrt{3}+2\sqrt{2})$
$= \sqrt{3}\sqrt{3} + \sqrt{3}\cdot 2\sqrt{2} + \sqrt{3}\cdot 2\sqrt{2} + 2\sqrt{2}\cdot 2\sqrt{2}$ — Distributive property
$= \sqrt{9} + 2\sqrt{6} + 2\sqrt{6} + 4\sqrt{4}$ — Multiply radicands
$= 3 + 2\sqrt{6} + 2\sqrt{6} + 4\cdot 2$ — Simplify radicals
$= 3 + 2\sqrt{6} + 2\sqrt{6} + 8$ — Multiply
$= 11 + 4\sqrt{6}$ — Combine like terms

▶ ***Quick check*** Simplify. $2(\sqrt{3}+\sqrt{5})$; $\sqrt{2}(\sqrt{14}+\sqrt{6})$; and $(\sqrt{2}+\sqrt{3})(\sqrt{2}-2\sqrt{3})$ ■

Conjugate factors

The type of factors that we are multiplying in examples 4 and 5 are called **conjugate factors.** The conjugate is used to rationalize the denominator of a fraction when the denominator contains two terms where one or both terms contain a square root. The idea of conjugate factors is derived from the factorization of the difference of two squares. When multiplying conjugate factors, we can simply write our answer as the square of the second term subtracted from the square of the first term.

In examples 4 and 5, we could have performed the multiplication as follows:

4. $(3 - \sqrt{2})(3 + \sqrt{2}) = (3)^2 - (\sqrt{2})^2 = 9 - 2 = 7$;
5. $(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$.

To determine what the conjugate of a given factor is, we write the original factor and change the sign of the second term.

■ *Example 9–5 B*

Form the conjugates of the given expressions.

1. $\sqrt{7} + 2$ — The conjugate is $\sqrt{7} - 2$.

2. $\sqrt{11} - \sqrt{6}$ — The conjugate is $\sqrt{11} + \sqrt{6}$.

3. $-5 - 2\sqrt{3}$ — The conjugate is $-5 + 2\sqrt{3}$.

4. $\sqrt{a} + \sqrt{b}$ — The conjugate is $\sqrt{a} - \sqrt{b}$.

▶ ***Quick check*** Form the conjugate. $6 - 3\sqrt{2}$ ■

Rationalizing the denominator

If we wish to rationalize the denominator of the fraction

$$\frac{1}{3 - \sqrt{2}}$$

we recall from example 9–5 A–4 that when we multiplied $3 - \sqrt{2}$ by $3 + \sqrt{2}$, there were no radicals left in our product. This result is precisely what we want to occur in our denominator. Therefore, to rationalize this fraction, we apply the fundamental principle of fractions and multiply the numerator and the denominator by $3 + \sqrt{2}$, the conjugate of the denominator.

$$\frac{1}{3 - \sqrt{2}} = \frac{1}{3 - \sqrt{2}} \cdot \frac{3 + \sqrt{2}}{3 + \sqrt{2}} \qquad 3 + \sqrt{2} \text{ is the conjugate of the denominator}$$

$$= \frac{1(3 + \sqrt{2})}{(3)^2 - (\sqrt{2})^2} \qquad (\text{first term})^2 - (\text{second term})^2$$

$$= \frac{3 + \sqrt{2}}{9 - 2} \qquad \text{No radicals remain in the denominator}$$

$$= \frac{3 + \sqrt{2}}{7} \qquad \text{Denominator is rationalized}$$

■ Example 9–5 C

Rationalize the denominators.

1. $$\frac{2}{\sqrt{7}+2} = \frac{2}{\sqrt{7}+2}\cdot\frac{\sqrt{7}-2}{\sqrt{7}-2}$$ Multiply by the conjugate of the denominator

$$= \frac{2(\sqrt{7}-2)}{(\sqrt{7})^2-(2)^2}$$ $(x+y)(x-y) = x^2 - y^2$

$$= \frac{2\sqrt{7}-4}{7-4}$$ Simplify in numerator and denominator

$$= \frac{2\sqrt{7}-4}{3}$$ Subtract in denominator

2. $$\frac{5}{\sqrt{11}-\sqrt{6}} = \frac{5}{\sqrt{11}-\sqrt{6}}\cdot\frac{\sqrt{11}+\sqrt{6}}{\sqrt{11}+\sqrt{6}}$$ Multiply by the conjugate of the denominator

$$= \frac{5(\sqrt{11}+\sqrt{6})}{(\sqrt{11})^2-(\sqrt{6})^2}$$ $(x+y)(x-y) = x^2 - y^2$

$$= \frac{5(\sqrt{11}+\sqrt{6})}{11-6}$$ Simplify radicals

$$= \frac{5(\sqrt{11}+\sqrt{6})}{5}$$ Subtract in denominator

$$= \sqrt{11}+\sqrt{6}$$ Reduce (by 5) to lowest terms

3. $$\frac{\sqrt{3}}{5-2\sqrt{3}} = \frac{\sqrt{3}}{5-2\sqrt{3}}\cdot\frac{5+2\sqrt{3}}{5+2\sqrt{3}}$$ Multiply by the conjugate of the denominator

$$= \frac{\sqrt{3}(5+2\sqrt{3})}{(5)^2-(2\sqrt{3})^2}$$ $(x+y)(x-y) = x^2 - y^2$

$$= \frac{5\sqrt{3}+2\sqrt{9}}{5^2-2^2(\sqrt{3})^2}$$ Simplify radicals

$$= \frac{5\sqrt{3}+2\cdot 3}{25-4\cdot 3}$$ Simplify radicals

$$= \frac{5\sqrt{3}+6}{25-12}$$ Perform operations

$$= \frac{5\sqrt{3}+6}{13}$$ Subtract in denominator

► ***Quick check*** Rationalize the denominators. $\frac{3}{7+\sqrt{2}}$ and $\frac{2}{\sqrt{11}-3}$ ■

Mastery points

Can you

- Multiply radical expressions containing more than one term?
- Form conjugate factors?
- Multiply conjugate factors?
- Rationalize a denominator that has two terms in which one or both terms contain a square root?

Exercise 9–5

Perform the indicated operations and simplify. Assume that all variables represent positive real numbers. See example 9–5 A.

Examples $2(\sqrt{3} + \sqrt{5})$

Solutions $= 2\sqrt{3} + 2\sqrt{5}$ Distributive property

$\sqrt{2}(\sqrt{14} + \sqrt{6})$

$= \sqrt{2}\sqrt{14} + \sqrt{2}\sqrt{6}$ Distributive property

$= \sqrt{28} + \sqrt{12}$ Product property

$= \sqrt{4 \cdot 7} + \sqrt{4 \cdot 3}$ Factor $28 = 4 \cdot 7$; $12 = 4 \cdot 3$

$= 2\sqrt{7} + 2\sqrt{3}$ $\sqrt{4} = 2$

Example $(\sqrt{2} + \sqrt{3})(\sqrt{2} - 2\sqrt{3})$

Solution $= \sqrt{2}\sqrt{2} - \sqrt{2} \cdot 2\sqrt{3} + \sqrt{3}\sqrt{2} - \sqrt{3} \cdot 2\sqrt{3}$ Distributive property

$= 2 - 2\sqrt{6} + \sqrt{6} - 2 \cdot 3$ $\sqrt{2}\sqrt{2} = 2$; $\sqrt{3}\sqrt{3} = 3$

$= 2 - \sqrt{6} - 6$ Combine like radicals

$= -4 - \sqrt{6}$ $2 - 6 = -4$

1. $3(\sqrt{2} + \sqrt{3})$
2. $5(2\sqrt{6} + \sqrt{2})$
3. $\sqrt{2}(\sqrt{3} + \sqrt{7})$
4. $\sqrt{5}(\sqrt{7} - \sqrt{3})$
5. $3\sqrt{2}(2\sqrt{3} - \sqrt{11})$
6. $\sqrt{6}(\sqrt{2} + \sqrt{3})$
7. $\sqrt{5}(\sqrt{15} - \sqrt{10})$
8. $\sqrt{14}(\sqrt{21} + \sqrt{10})$
9. $2\sqrt{7}(\sqrt{35} - 3\sqrt{14})$
10. $\sqrt{a}(\sqrt{ab} + \sqrt{b})$
11. $\sqrt{a}(3\sqrt{a} + \sqrt{b})$
12. $(5 + \sqrt{3})(4 - \sqrt{3})$
13. $(3 + \sqrt{2})(4 + \sqrt{2})$
14. $(5 - \sqrt{5})(5 - \sqrt{5})$
15. $(3 - 4\sqrt{a})(4 - 3\sqrt{a})$
16. $(7 + 2\sqrt{y})(6 + 5\sqrt{y})$
17. $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$
18. $(\sqrt{7} + \sqrt{5})(\sqrt{7} - \sqrt{5})$
19. $(2 + \sqrt{6})(2 - \sqrt{6})$
20. $(5 - \sqrt{3})(5 + \sqrt{3})$
21. $(2 + \sqrt{5})^2$
22. $(3 - \sqrt{7})^2$
23. $(\sqrt{x} + \sqrt{y})^2$
24. $(\sqrt{a} - \sqrt{b})^2$
25. $(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})$
26. $(2\sqrt{a} - \sqrt{b})(2\sqrt{a} + \sqrt{b})$
27. $(x\sqrt{y} + \sqrt{z})(x\sqrt{y} - \sqrt{z})$
28. $(a\sqrt{b} + c)(a\sqrt{b} - c)$
29. $(2\sqrt{x} + y)^2$
30. $(3\sqrt{a} + \sqrt{b})^2$

Form the conjugate of the given expressions. See example 9–5 B.

Example $6 - 3\sqrt{2}$

Solution $6 + 3\sqrt{2}$ First term remains the same, change the sign of the second term

31. $11 - \sqrt{3}$
32. $-5\sqrt{7} - \sqrt{2}$
33. $\sqrt{a} + 3\sqrt{b}$
34. $a\sqrt{b} - \sqrt{c}$

Rationalize the denominators. Assume that all variables represent positive real numbers and that no denominator is equal to zero. See example 9–5 C.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Simplify the following expressions, leaving all denominators rationalized. Assume that all variables represent positive real numbers and that no denominator is equal to zero. See example 9–5 C.

Examples $\dfrac{3}{7+\sqrt{2}}$

Solutions

$$= \frac{3}{7+\sqrt{2}} \cdot \frac{7-\sqrt{2}}{7-\sqrt{2}}$$ Multiply by the conjugate

$$= \frac{3(7-\sqrt{2})}{(7)^2-(\sqrt{2})^2}$$ $(x + y)(x - y) = x^2 - y^2$

$$= \frac{3(7-\sqrt{2})}{49-2}$$ Simplify denominator

$$= \frac{3(7-\sqrt{2})}{47}$$ Simplify denominator

$$= \frac{21-3\sqrt{2}}{47}$$ Multiply in numerator

Examples $\dfrac{2}{\sqrt{11}-3}$

Solutions

$$= \frac{2}{\sqrt{11}-3} \cdot \frac{\sqrt{11}+3}{\sqrt{11}+3}$$ Multiply by the conjugate

$$= \frac{2(\sqrt{11}+3)}{(\sqrt{11})^2-(3)^2}$$ $(x + y)(x - y) = x^2 - y^2$

$$= \frac{2(\sqrt{11}+3)}{11-9}$$ Simplify denominator

$$= \frac{2(\sqrt{11}+3)}{2}$$ Simplify denominator

$$= \sqrt{11}+3$$ Reduce fraction

35. $\dfrac{1}{\sqrt{2}+3}$ **36.** $\dfrac{1}{\sqrt{3}-2}$ **37.** $\dfrac{7}{2+\sqrt{7}}$ $\boxed{\textbf{38.}}$ $\dfrac{6}{3-\sqrt{6}}$

39. $\dfrac{3}{\sqrt{6}-\sqrt{3}}$ **40.** $\dfrac{1}{\sqrt{a}+b}$ **41.** $\dfrac{3}{2\sqrt{3}-\sqrt{5}}$ **42.** $\dfrac{4}{2\sqrt{3}-\sqrt{6}}$

$\boxed{\textbf{43.}}$ $\dfrac{1+\sqrt{5}}{1-\sqrt{5}}$ **44.** $\dfrac{\sqrt{3}-\sqrt{7}}{\sqrt{3}+\sqrt{7}}$ **45.** $\dfrac{\sqrt{a}+b}{\sqrt{a}-b}$

Review exercises

Perform the indicated operations and leave your answer with only positive exponents. Assume that no variable is equal to zero. See sections 3–1 and 3–3.

1. $(2^2)^3$ **2.** $2^2 \cdot 2^3$ **3.** $3^{-2} \cdot 3^4$ **4.** $\dfrac{3^2}{3^5}$

5. $(x^{-2})^3$ **6.** $(2a^2b)^3$ **7.** $\dfrac{x^2y^5}{x^3y^2}$ **8.** $x^{-4} \cdot x^{-5}$

9–6 ■ Fractional exponents

Fractional exponents

In this section, we are going to develop the idea of a fraction used as an exponent. Consider the example

$$(a^{1/2})^2 = a^{1/2 \cdot 2} = a^1$$

When we raise a power to a power, we multiply the exponents. In the previous section, we observed that if a represented only nonnegative real numbers, then

$$(\sqrt{a})^2 = a$$

Therefore, for our properties of exponents and our procedures for radicals to be consistent, the following statement must be true:

$$a^{1/2} = \sqrt{a}$$

We generalize as follows:

Definition of $a^{1/n}$

$$a^{1/n} = \sqrt[n]{a}$$

where n is a natural number greater than 1. Whenever n is even, a represents only nonnegative real numbers.

Concept

The expression $a^{1/n}$ represents the principal nth root of a.

■ **Example 9–6 A**

Rewrite the following in radical notation and simplify where possible.

1. $5^{1/2} = \sqrt{5}$

2. $(64)^{1/2} = \sqrt{64} = 8$

3. $a^{1/3} = \sqrt[3]{a}$

4. $(-8)^{1/3} = \sqrt[3]{(-8)} = -2$ ■

Once we become acquainted with fractional exponents, the process of changing the fractional exponent to radical form for simplification will become unnecessary.

Consider the following problem:

$$(\sqrt[3]{a})^2 = (a^{1/3})^2 = a^{2/3}$$

We observe that *when a number is raised to a fractional exponent, the numerator of the fractional exponent indicates the power to which the base is to be raised, and the denominator indicates the root to be taken.*

Definition of $a^{m/n}$

If a is any real number, m is any integer, and n is any positive integer, then if $\frac{m}{n}$ is reduced to lowest terms,

$$a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

provided that $\sqrt[n]{a}$ is a real number.

Concept

The expression $a^{m/n}$ represents the principal nth root of a raised to the mth power.

We calculate $a^{m/n}$ *by first finding the principal* n*th root of* a *and then raising the resulting number to the* m*th power.*

■ ***Example 9–6 B***

Simplify. Assume that all variables represent nonnegative real numbers.

Numerator is the power — Denominator is the index

1. $(16)^{3/4} = (\sqrt[4]{16})^3$
$= 2^3$ 4th root of 16 is 2
$= 8$ Standard form

2. $(32)^{2/5} = (\sqrt[5]{32})^2 = 2^2 = 4$

3. $(-27)^{2/3} = (\sqrt[3]{-27})^2 = (-3)^2 = 9$

4. $(x^6)^{1/3} = \sqrt[3]{x^6} = x^2$

▶ ***Quick check*** Simplify. $(49)^{1/2}$ and $(8)^{2/3}$ ■

Operations with fractional exponents

We can extend the properties and definitions involving integer exponents to expressions that involve fractional exponents.

■ ***Example 9–6 C***

Perform the indicated operations and simplify. Assume that all variables represent positive real numbers.

1. $5^{1/2} \cdot 5^{1/3} = 5^{1/2 + 1/3}$ Multiply like bases, add exponents
$= 5^{3/6 + 2/6}$ Least common denominator is 6
$= 5^{5/6}$ Add numerators

2. $\dfrac{2^{1/2}}{2^{1/4}} = 2^{1/2 - 1/4} = 2^{2/4 - 1/4} = 2^{1/4}$

3. $(-27)^{-2/3} = \dfrac{1}{(-27)^{2/3}} = \dfrac{1}{(\sqrt[3]{-27})^2} = \dfrac{1}{(-3)^2} = \dfrac{1}{9}$

4. $(2^3x^9y^{15})^{1/3} = (2^3)^{1/3}(x^9)^{1/3}(y^{15})^{1/3} = 2^{3 \cdot 1/3} \cdot x^{9 \cdot 1/3} \cdot y^{15 \cdot 1/3} = 2^1x^3y^5$
$= 2x^3y^5$

5. $x^{2/3} \cdot x^{3/4} = x^{2/3 + 3/4} = x^{8/12 + 9/12} = x^{17/12}$

6. $(y^{1/2})^{4/3} = y^{1/2 \cdot 4/3} = y^{2/3}$

7. $\dfrac{z^{1/2}}{z^{2/3}} = z^{1/2 - 2/3} = z^{3/6 - 4/6} = z^{-1/6} = \dfrac{1}{z^{1/6}}$

Note In example 7, $z^{1/6}$ in the denominator is simply another form of $\sqrt[6]{z}$. Therefore, if we want our answer in a rationalized form, we would proceed as follows:

$$\frac{1}{z^{1/6}} = \frac{1}{\sqrt[6]{z}} \cdot \frac{\sqrt[6]{z^5}}{\sqrt[6]{z^5}} = \frac{\sqrt[6]{z^5}}{z} = \frac{z^{5/6}}{z}$$

▶ ***Quick check*** Simplify. $y^{2/3} \cdot y^{3/2}$; $\dfrac{a^{3/4}b^{5/3}}{a^{1/4}b}$; and $(36)^{-1/2}$ ■

Mastery points

Can you

- Express fractional exponents in radical form?
- Express radicals in fractional exponent form?
- Apply the properties and definitions involving integer exponents to fractional exponents?

Exercise 9–6

Simplify the given expressions. See examples 9–6 A and B.

Examples $(49)^{1/2}$

Solutions $= \sqrt{49}$ Square root
$= 7$

$8^{2/3}$
$= (\sqrt[3]{8})^2$ Numerator is the power, Denominator is the index
$= (2)^2$ Cube root of 8 is 2
$= 4$ Standard form

$(36)^{-1/2}$
$= \dfrac{1}{(36)^{1/2}}$ $a^{-n} = \dfrac{1}{a^n}$
$= \dfrac{1}{\sqrt{36}}$ Square root
$= \dfrac{1}{6}$ Simplify

1. $(36)^{1/2}$ 2. $(25)^{1/2}$ $\boxed{3.}$ $(a^6)^{1/3}$ 4. $(b^{12})^{1/3}$ 5. $(8)^{1/3}$ 6. $(32)^{1/5}$
$\boxed{7.}$ $(-27)^{1/3}$ 8. $(-8)^{1/3}$ $\boxed{9.}$ $(27)^{2/3}$ 10. $(16)^{3/4}$ 11. $(9)^{3/2}$ 12. $(16)^{3/2}$

Perform the indicated operations and simplify. Assume that all variables represent positive real numbers. See example 9–6 C.

Examples $y^{2/3} \cdot y^{3/2}$

Solutions $= y^{2/3 + 3/2}$ Multiply like bases, add exponents
$= y^{4/6 + 9/6}$ Least common denominator is 6
$= y^{13/6}$ Add numerators

$\dfrac{a^{3/4}b^{5/3}}{a^{1/4}b}$
$= a^{3/4 - 1/4}b^{5/3 - 1}$ Divide like bases, subtract exponents
$= a^{2/4}b^{5/3 - 3/3}$ Common denominators
$= a^{1/2}b^{2/3}$ Subtract numerators

13. $(25)^{-1/2}$ 14. $(9)^{-1/2}$ 15. $(16)^{-3/4}$ 16. $(27)^{-2/3}$ $\boxed{17.}$ $(-8)^{-1/3}$ 18. $(-27)^{-1/3}$
19. $2^{1/2} \cdot 2^{3/2}$ 20. $3^{1/3} \cdot 3^{2/3}$ 21. $2^{1/3} \cdot 2^{1/2}$ 22. $5^{1/5} \cdot 5^{1/2}$ 23. $x^{1/4} \cdot x^{3/4}$ 24. $b^{1/3} \cdot b^{2/3}$
25. $c^{1/2} \cdot c^{1/4}$ 26. $x^{1/4} \cdot x^{1/3}$ 27. $\dfrac{2^{3/2}}{2^{1/2}}$ 28. $\dfrac{3^{4/3}}{3^{1/3}}$ 29. $\dfrac{2^{1/2}}{2^{1/3}}$ 30. $\dfrac{7^{3/4}}{7^{2/3}}$
31. $\dfrac{a^{4/5}}{a^{1/5}}$ 32. $\dfrac{x^{3/4}}{x^{1/2}}$ 33. $\dfrac{y^{2/3}}{y^{1/2}}$ 34. $\dfrac{x^{5/6}}{x^{2/3}}$ $\boxed{35.}$ $(a^{2/3})^{1/2}$ 36. $(c^{1/2})^{1/2}$
37. $(x^{1/2})^{4/3}$ 38. $(y^{2/3})^{3/4}$ $\boxed{39.}$ $(c^{-1/4})^{-2/3}$ 40. $(y^{-1/2})^{-2/5}$ 41. $(a^{-2/3})^{-1/2}$ 42. $(b^{-1/2})^{-1/2}$
43. $(x^{1/4})^{-2/3}$ 44. $(x^{1/3})^{-3/4}$ 45. $(y^{-3/4})^{1/3}$ 46. $(c^{-2/5})^{1/2}$ 47. $(16a^4)^{3/4}$ 48. $(x^3y^{12})^{1/3}$
49. $(8a^6b^3)^{2/3}$ 50. $(27x^3y^{12})^{2/3}$ $\boxed{51.}$ $\dfrac{b^{3/4}c^{1/2}}{b^{1/4}c^{1/4}}$ 52. $\dfrac{xy^{3/4}}{x^{1/2}y^{1/4}}$ 53. $\dfrac{ab}{a^{1/2}b^{1/3}}$ 54. $\dfrac{xy^{3/4}}{x^{2/5}y^{1/2}}$

Solve the following word problems.

55. Find the number whose principal fourth root is 3.

56. Find the number whose principal cube root is -2.

57. The formula for approximating the velocity V in miles per hour of a car based on the length of its skid marks S (in feet) on dry pavement is given by

$V = 4.9\ S^{1/2}$

If the skid marks are 100 feet long, what was the approximate velocity?

58. The formula for approximating the velocity V in miles per hour of a car based on the length of its skid marks S (in feet) on wet pavement is given by

$V = 3.46\ S^{1/2}$

If the skid marks are 81 feet long, what was the approximate velocity of the car?

59. At an altitude of h feet above the sea or level ground, the distance d in miles that a person can see an object is found by using the equation

$d = 1.2\ h^{1/2}$

How far can someone see who is in a tower 400 feet above the ground?

60. How can you find the principal fourth root of a number on a calculator using only the square root key?

61. How can you find the principal eighth root of a number on a calculator using only the square root key?

Review exercises

Perform the indicated operations. Assume that all radicands are nonnegative. See sections 9–1 and 9–2.

1. $(\sqrt{7})^2$ **2.** $(\sqrt{x})^2$ **3.** $(\sqrt{x+1})^2$

Perform the indicated operations. See section 3–2.

4. $(x + 1)^2$ **5.** $(x - 2)^2$

Find the solution set for the following equations. See section 4–7.

6. $x + 6 = x^2$ **7.** $x + 2 = x^2 - 9x + 18$ **8.** $x + 1 = x^2 + 2x + 1$

9–7 ■ Equations involving radicals

Radical equations

An equation in which the unknown quantity appears under a radical symbol is called a **radical equation.** Examples of radical equations are

$$\sqrt{x} = 5; \quad \sqrt{x+2} = 7; \quad 4 + \sqrt{x+2} = x$$

In this section, we will consider radical equations containing only square roots.

Solving radical equations involves squaring both members of an equation to eliminate the square roots. When we square both members of an equation, we use the squaring property of equality.

Squaring property of equality

If P and Q are algebraic expressions and if

$$P = Q$$

then all solutions of $P = Q$ are also solutions of the equation

$$P^2 = Q^2$$

Concept

If each member of an equation is squared, the solution(s) of the original equation are solution(s) of the resulting equation.

Extraneous solutions

This property implies that there *may be* solutions of the equation $P^2 = Q^2$ that are not solutions of the original equation $P = Q$. If such solutions exist, they are called **extraneous solutions** (roots). Thus, all possible solutions must be checked in the original equation.

To solve a radical equation

1. Rewrite the equation (if necessary) so that a radical is by itself in one member of the equation.
2. Square each member (side) of the equation and combine like terms.
3. Repeat steps 1 and 2 if a radical remains in the equation.
4. Solve the resulting equation.
5. Check all possible solutions in the original equation.

■ Example 9–7 A

Find the solution set.

1. $\sqrt{x} = 5$

$(\sqrt{x})^2 = (5)^2$ — Square both members

$x = 25$ — Squaring a square root gives the radicand

Check:

$\sqrt{x} = 5$ — Original equation

$\sqrt{(25)} = 5$ — Substitute into original equation

$5 = 5$ — True

The solution set is $\{25\}$.

2. $\sqrt{x + 2} = 7$

$(\sqrt{x + 2})^2 = (7)^2$ — Square both members

$x + 2 = 49$ — Squaring a square root gives the radicand

$x = 47$ — Subtract 2 from both members

Check:

$\sqrt{x + 2} = 7$ — Original equation

$\sqrt{(47) + 2} = 7$ — Substitute into original equation

$\sqrt{49} = 7$ — Simplify the radicand

$7 = 7$ — True

The solution set is $\{47\}$.

3. $4 + \sqrt{x + 2} = x$

$\sqrt{x + 2} = x - 4$	Isolate the radical by subtracting 4
$(\sqrt{x + 2})^2 = (x - 4)^2$	Square both members
$x + 2 = (x - 4)(x - 4)$	Simplify
$x + 2 = x^2 - 4x - 4x + 16$	Multiply
$x + 2 = x^2 - 8x + 16$	Subtract x and 2
$0 = x^2 - 9x + 14$	Solve the resulting quadratic equation
$0 = (x - 7)(x - 2)$	Factor
$x - 7 = 0$ or $x - 2 = 0$	Set each factor equal to 0 and solve
$x = 7$ or $x = 2$	Check for extraneous roots

Check:

for 7	for 2	
$4 + \sqrt{x + 2} = x$	$4 + \sqrt{x + 2} = x$	Original equation
$4 + \sqrt{(7) + 2} = (7)$	$4 + \sqrt{(2) + 2} = (2)$	Substitute
$4 + \sqrt{9} = 7$	$4 + \sqrt{4} = 2$	Simplify
$4 + 3 = 7$	$4 + 2 = 2$	Simplify
$7 = 7$ True	$6 = 2$	False

Therefore, 7 is the only solution to the equation. 2 is an *extraneous root* and the solution set is $\{7\}$.

4. $\sqrt{x + 1} = x + 1$

$(\sqrt{x + 1})^2 = (x + 1)^2$	Square both members
$x + 1 = (x + 1)(x + 1)$	Simplify
$x + 1 = x^2 + x + x + 1$	Multiply
$x + 1 = x^2 + 2x + 1$	Combine like terms
$0 = x^2 + x$	Solve the resulting quadratic equation
$0 = x(x + 1)$	Factor
$x = 0$ or $x + 1 = 0$	Set each factor equal to 0 and solve

Therefore, $x = 0$ or $x = -1$.

Check:

for 0	for -1	
$\sqrt{x + 1} = x + 1$	$\sqrt{x + 1} = x + 1$	Original equation
$\sqrt{(0) + 1} = (0) + 1$	$\sqrt{(-1) + 1} = (-1) + 1$	Substitute
$\sqrt{1} = 1$	$\sqrt{0} = 0$	Simplify
$1 = 1$ True	$0 = 0$	True

Therefore, 0 and -1 are solutions of the equation. The solution set is $\{0, -1\}$.

5. $\sqrt{3x + 4} = \sqrt{x + 14}$

Note There are two square roots in this problem, but we can eliminate both radical symbols by squaring both members.

$(\sqrt{3x + 4})^2 = (\sqrt{x + 14})^2$	Square both members
$3x + 4 = x + 14$	Solve for x
$2x + 4 = 14$	Subtract x from both members
$2x = 10$	Subtract 4 from both members
$x = 5$	Divide both members by 2

Check:

$\sqrt{3x + 4} = \sqrt{x + 14}$	Original equation
$\sqrt{3(5) + 4} = \sqrt{(5) + 14}$	Substitute
$\sqrt{15 + 4} = \sqrt{19}$	Simplify
$\sqrt{19} = \sqrt{19}$	True

The solution set is $\{5\}$.

6. $\sqrt{x+3} = -6$

$$\begin{aligned} (\sqrt{x+3})^2 &= (-6)^2 && \text{Square both members} \\ x + 3 &= 36 && \text{Solve for } x \\ x &= 33 && \text{Subtract 3 from both members} \end{aligned}$$

Check:

$$\begin{aligned} \sqrt{x+3} &= -6 && \text{Original equation} \\ \sqrt{(33)+3} &= -6 && \text{Substitute} \\ \sqrt{36} &= -6 && \text{Simplify} \\ 6 &= -6 && \text{False} \end{aligned}$$

$x = 33$ does not check because $\sqrt{36} = 6$, not -6. We conclude that there is *no solution* to this equation, and the solution set is $\emptyset$.

▶ ***Quick check*** Find the solution set. $\sqrt{x+5} = 6$ and $\sqrt{2x+5} = \sqrt{x+8}$ ■

Mastery points

Can you

- Solve equations containing radicals?

Exercise 9–7

Find the solution set. See example 9–7 A.

Examples $\sqrt{x+5} = 6$

Solutions

$$\begin{aligned} (\sqrt{x+5})^2 &= (6)^2 \\ x + 5 &= 36 \\ x &= 31 \end{aligned}$$

Check:

$$\begin{aligned} \sqrt{(31)+5} &= 6 \\ \sqrt{36} &= 6 \\ 6 &= 6 && \text{True} \end{aligned}$$

The solution set is $\{31\}$.

$\sqrt{2x+5} = \sqrt{x+8}$

$$\begin{aligned} (\sqrt{2x+5})^2 &= (\sqrt{x+8})^2 && \text{Square both members} \\ 2x + 5 &= x + 8 && \text{Simplify} \\ x + 5 &= 8 && \text{Solve for } x \\ x &= 3 \end{aligned}$$

Check:

$$\begin{aligned} \sqrt{2(3)+5} &= \sqrt{(3)+8} && \text{Substitute into original equation} \\ \sqrt{6+5} &= \sqrt{11} && \text{Simplify} \\ \sqrt{11} &= \sqrt{11} && \text{True} \end{aligned}$$

The solution set is $\{3\}$.

1. $\sqrt{x} = 4$
2. $\sqrt{x} = 5$
3. $\sqrt{x} = 9$
4. $\sqrt{x} = 7$
5. $\sqrt{x+5} = 4$
6. $\sqrt{x-3} = 5$
7. $\sqrt{x-7} = 6$
8. $\sqrt{x+3} = 7$
9. $\sqrt{2x+1} = 5$
10. $\sqrt{2x+6} = 6$
11. $\sqrt{3x+1} = 4$
12. $\sqrt{5x-4} = 6$
13. $\sqrt{x} + 4 = 7$
14. $\sqrt{x} + 2 = 9$
15. $\sqrt{x} - 5 = 1$
16. $\sqrt{x} - 4 = 2$
17. $\sqrt{x} + 7 = 5$
18. $\sqrt{x} + 8 = 4$
19. $\sqrt{x} + 6 = 3$
20. $\sqrt{x} + 10 = 5$
21. $\sqrt{2x+1} = \sqrt{x+5}$
22. $\sqrt{2x+4} = \sqrt{3x-2}$
23. $\sqrt{5x-3} = \sqrt{2x+9}$
24. $\sqrt{7x-4} = \sqrt{3x+20}$
25. $\sqrt{2x+7} = \sqrt{4x+1}$
26. $\sqrt{6x-3} = \sqrt{4x+5}$
27. $\sqrt{3x+5} = \sqrt{5x+1}$
28. $\sqrt{4x-4} = \sqrt{x+5}$
29. $\sqrt{5x-7} = \sqrt{2x+8}$
30. $\sqrt{4x-15} = \sqrt{7x-9}$
31. $\sqrt{5+2x} = \sqrt{2+3x}$
32. $\sqrt{9-5x} = \sqrt{15-7x}$

33. $\sqrt{x}\sqrt{x-15}=4$
34. $\sqrt{x}\sqrt{x-3}=2$
35. $\sqrt{x}\sqrt{x-8}=3$
36. $\sqrt{x}\sqrt{x-6}=4$
37. $\sqrt{x}\sqrt{x+6}=4$
38. $\sqrt{x+3}\sqrt{x}=2$
39. $\sqrt{x^2+1}=x+2$
40. $\sqrt{x^2+3x}=x+1$
41. $\sqrt{x^2+3x}=x-3$
42. $\sqrt{x^2+12}=x+2$
43. $\sqrt{x+6}=x$
44. $\sqrt{5x+6}=x$
45. $\sqrt{2x+8}=x$
46. $\sqrt{4x+12}=x$
47. $\sqrt{x-4}=x-6$
48. $\sqrt{x+2}=x+2$
49. $\sqrt{x+4}+8=x$
50. $\sqrt{x}+6=x$
51. $\sqrt{x+7}=2x-1$
52. $\sqrt{2x-1}+2x=7$

Find the unknown number in problems 53–60.

53. The square root of the sum of a number and 6 is 5. Find the number.

54. The square root of the sum of a number and 9 is 7. Find the number.

55. The square root of the product of a number and 6 is 12. Find the number.

56. The square root of the product of a number and 9 is 6. Find the number.

57. A certain number is equal to the square root of the sum of that number and 12.

58. The square root of the product of a number and 12 is equal to the number increased by 3.

59. The square root of the sum of a number and 11 is 1 less than the number.

60. The square root of the product of a number and 4 is 3 less than the number.

61. At an altitude of h ft above the sea or level ground, the distance d in miles that a person can see an object is given by

$$d=\sqrt{\frac{3h}{2}}$$

How high must a person be to see an object 6 miles away?

62. The formula for approximating the velocity V in miles per hour of a car based on the length of its skid marks S (in feet) on dry pavement is given by

$$V=2\sqrt{6S}$$

If the velocity is 36 mph, how long will the skid marks be?

63. On wet pavement, the formula in exercise 62 is given by

$$V=2\sqrt{3S}$$

How long will the skid marks be if the car is traveling at 36 mph on wet pavement?

64. Find the number whose principal square root is 3. (*Hint:* Let $\sqrt{x}=3$)

65. Find the number whose principal square root is 10.

66. Find the number whose principal square root is 11.

Review exercises

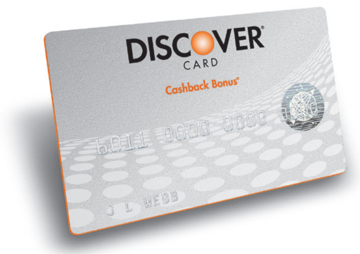
Completely factor the following expressions. See sections 4–2 and 4–4.

1. x^2-4
2. $x^2+9x+18$
3. $x^2-3x-10$
4. x^2-6x+9

Simplify. See section 9–1.

5. $\sqrt{81}$
6. $\sqrt{49}$
7. $\sqrt{121}$

8. Find the solution set for the equation $x^2=64$. See section 4–7.



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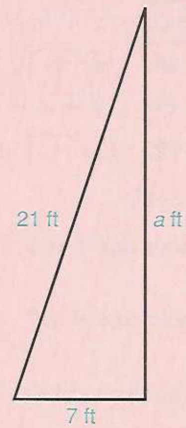
Chapter 9 lead-in problem

Roger has a ladder that will extend to a length of 21 feet. For the ladder to be safe to climb on, it must be placed 7 feet away from the house. The roof is 20 feet above the ground. Will the ladder be able to reach the roof safely? If not, how far up will the ladder reach? (Leave your answer rounded to one decimal place.)

Solution



$a^2 + b^2 = c^2$	Pythagorean Theorem
$a^2 + (7)^2 = (21)^2$	Substitute 21 for c and 7 for b
$a^2 + 49 = 441$	Simplify
$a^2 = 392$	Isolate a^2
$a \approx 19.8$	Round $\sqrt{392}$ to one decimal place

The ladder will not be able to reach the roof safely. The ladder's maximum safe reach is approximately 19.8 feet.

Chapter 9 summary

1. $\sqrt[n]{a} = b$ if $\overbrace{b \cdot b \cdot b \cdots b}^{n \text{ factors}} = b^n = a$, where n is a natural number greater than 1.
2. If we exclude even roots of negative numbers, which do not exist in the set of real numbers, the **principal *n*th root** of a number, denoted by $\sqrt[n]{}$, has the same sign as the number itself.
3. When we multiply two radicals *having the same index,* we multiply the radicands and put the product under a radical symbol with the common index. $\sqrt{a}\sqrt{b} = \sqrt{ab}$ and $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$, **product property for radicals.**
4. We can simplify a radical if the radicand has a factor(s) whose exponent is equal to or greater than the value of the index.
5. The nth root of a *fraction* can be written as the nth root of the numerator over the nth root of the denominator. $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$ and $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$, where $b \neq 0$, **quotient property for radicals.**
6. We eliminate radicals from the denominator of a fraction by **rationalizing** the denominator.
7. We can only add or subtract **like radicals.**
8. **Conjugate factors** are used to rationalize a denominator when the denominator has two terms where one or both terms contain a square root.
9. When a number is raised to a **fractional exponent,** the numerator of the fractional exponent indicates the power to which the base is to be raised, and the denominator indicates the root to be taken.
10. If a is any real number, m is any integer, and n is any positive integer, then if $\dfrac{m}{n}$ is reduced to lowest terms, $a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$ provided that $\sqrt[n]{a}$ is a real number.
11. The **squaring property of equality** states that if both members of an equation are squared, the solution(s) of the original equation are solution(s) of the resulting equation.
12. When solving equations where the unknown is under a radical symbol, we must check for **extraneous roots.**

Chapter 9 error analysis

1. Principal square root
Example: $\sqrt{36} = -6$
Correct answer: 6
What error was made? (*see page 368*)
2. Rationalizing the denominator of an nth root
Example: $\dfrac{x}{\sqrt[3]{x}} = \dfrac{x}{\sqrt[3]{x}} \cdot \dfrac{\sqrt[3]{x}}{\sqrt[3]{x}} = \dfrac{x\sqrt[3]{x}}{x} = \sqrt[3]{x}$
Correct answer: $\sqrt[3]{x^2}$
What error was made? (*see page 380*)
3. Addition in the radicand of a radical expression
Example: $\sqrt{9+4} = \sqrt{9} + \sqrt{4} = 3 + 2 = 5$
Correct answer: $\sqrt{13}$
What error was made? (*see page 374*)
4. Subtraction of radical expressions
Example: $\sqrt{3} - \sqrt{2} = \sqrt{3-2} = \sqrt{1} = 1$
Correct answer: $\sqrt{3} - \sqrt{2}$
What error was made? (*see page 374*)
5. Sums and differences of radical numbers
Example:
$\sqrt{8} + 3\sqrt{2} - \sqrt{18} = 2\sqrt{2} + 3\sqrt{2} - \sqrt{18}$
$= 5\sqrt{2} - \sqrt{18}$
Correct answer: $2\sqrt{2}$
What error was made? (*see page 384*)
6. Multiplication of radical expressions
Example: $\sqrt{3}(\sqrt{3} + \sqrt{2}) = \sqrt{3} \cdot \sqrt{3} + \sqrt{2} = 3 + \sqrt{2}$
Correct answer: $3 + \sqrt{6}$
What error was made? (*see page 387*)
7. Squaring a radical binomial
Example: $(\sqrt{2} - \sqrt{3})^2 = (\sqrt{2})^2 - (\sqrt{3})^2 = 2 - 3$
$= -1$
Correct answer: $5 - 2\sqrt{6}$
What error was made? (*see page 387*)
8. Fractional exponents to radical expressions
Example: $x^{3/2} = \sqrt[3]{x^2}$
Correct answer: $x\sqrt{x}$
What error was made? (*see page 392*)
9. Extraneous solutions of radical equations
Example: Find the solution set of $\sqrt{x+2} = x - 4$
$(\sqrt{x+2})^2 = (x-4)^2$
$x + 2 = x^2 - 8x + 16$
$x^2 - 9x + 14 = 0$
$(x-7)(x-2) = 0$
$x - 7 = 0$ or $x - 2 = 0$
$x = 7$ or $x = 2$ $\quad \{2,7\}$
Correct answer: $\{7\}$
What error was made? (*see page 396*)
10. Negative exponents
Example: $6^{-2} = -36$
Correct answer: $\dfrac{1}{36}$
What error was made? (*see page 141*)

Chapter 9 critical thinking

Given the numbers 33 and 27, determine a method by which you can multiply these numbers mentally.

Chapter 9 review

Assume that all variables in problems 1–42 represent positive real numbers and that no denominator is equal to zero.

[9–1]

Find the indicated root.

1. $\sqrt{81}$
2. $\sqrt{25}$
3. $-\sqrt{9}$
4. $-\sqrt{49}$

[9–2]

Perform any indicated operations and simplify.

5. $\sqrt{40}$
6. $\sqrt{18a^2b^3}$
7. $\sqrt{2}\sqrt{14}$
8. $\sqrt{18}\sqrt{10}$

[9–3]

Express the given radicals in simplest form with all denominators rationalized.

9. $\sqrt{\dfrac{16}{17}}$
10. $\sqrt{\dfrac{7}{18}}$
11. $\sqrt{\dfrac{a}{b}}$
12. $\sqrt{\dfrac{x}{y^3}}$
13. $\dfrac{a}{\sqrt{ab}}$
14. $\dfrac{2x}{\sqrt{xy}}$

[9–4]

Perform the indicated operations and simplify.

15. $3\sqrt{7} + 4\sqrt{7}$
16. $\sqrt{18} + 5\sqrt{2}$
17. $3\sqrt{20} - \sqrt{45}$
18. $2\sqrt{75} - \sqrt{3} + 5\sqrt{27}$
19. $\sqrt{50a} - 2\sqrt{8a}$
20. $7\sqrt{9x} - 2\sqrt{4x}$

[9–5]

Perform the indicated operations and simplify.

21. $\sqrt{3}(\sqrt{5} - \sqrt{7})$
22. $\sqrt{10}(\sqrt{14} + \sqrt{6})$
23. $(5 + \sqrt{7})(3 - \sqrt{7})$
24. $(6 - \sqrt{3})^2$
25. $(\sqrt{3} + \sqrt{5})^2$
26. $(2\sqrt{a} - \sqrt{b})(2\sqrt{a} + \sqrt{b})$

Express the given radicals in simplest form with all denominators rationalized.

27. $\dfrac{1}{\sqrt{3} - 2}$
28. $\dfrac{2}{\sqrt{6} + 4}$
29. $\dfrac{1}{\sqrt{a} + b}$
30. $\dfrac{x}{\sqrt{xy} + x}$
31. $\dfrac{a}{a + \sqrt{b}}$
32. $\dfrac{\sqrt{2} + 3}{\sqrt{2} - 3}$

[9–6]

Simplify the given expressions.

33. $(36)^{1/2}$
34. $(8)^{2/3}$
35. $(-8)^{1/3}$
36. $(32)^{-2/5}$

Perform the indicated operations and simplify.

37. $a^{2/5} \cdot a^{3/5}$
38. $b^{1/3} \cdot b^{3/4}$
39. $\dfrac{a^{1/2}}{a^{1/4}}$
40. $(a^{3/2})^{3/4}$
41. $(16a^4b^8)^{3/4}$
42. $\dfrac{a^2b^2}{a^{1/2}b^{3/2}}$

[9–7]

Find the solution set.

43. $\sqrt{x} = 8$
44. $\sqrt{x - 4} = 7$
45. $\sqrt{5x - 3} = \sqrt{3x + 5}$
46. $\sqrt{x}\sqrt{x + 8} = 3$
47. $\sqrt{x}\sqrt{x + 6} = 4$
48. $\sqrt{x^2 + 16} = x + 2$
49. $\sqrt{x + 6} = x + 4$
50. $\sqrt{x - 3} = x - 3$

Chapter 9 cumulative test

Perform the indicated operations and simplify. Assume that all variables represent positive real numbers and that no denominator is equal to zero.

[1–8] 1. $3[4(6 - 2) + (-5 + 4)]$
[3–1] 2. $x^3 \cdot x^2 \cdot x^2$
[3–3] 3. $x^{12} \div x^6$
[2–3] 4. $(3x^2 - 2x + 5) - (x^2 - 4x - 8)$
[3–2] 5. $4a^2b^3(2a^2 - 3ab + 4b^2)$
[3–3] 6. $\dfrac{8a^{-2}b^3c^0}{4a^{-5}b}$
[1–8] 7. -6^2
[3–2] 8. $(3a - b)^2$
[6–1] 9. $\dfrac{a^2 - 9}{3a + 12} \cdot \dfrac{a + 4}{a^2 + 5a + 6}$
[9–4] 10. $5\sqrt{12} + 2\sqrt{27}$
[9–1] 11. $\sqrt[3]{-8}$
[6–2] 12. $\dfrac{2x}{x - 1} - \dfrac{x - 3}{x - 1}$
[9–5] 13. $\dfrac{\sqrt{x}}{x - \sqrt{y}}$
[9–2] 14. $\sqrt[3]{81x^4y^6z}$
[3–2] 15. $(5x - y)(5x + y)$
[2–3] 16. $3x - [2x - (x - y) + 3y]$

Factor completely.

[4–1] **17.** $6a^3b^4 - 2a^3b^5 + 8a^5b^3$

[4–4] **18.** $25c^2 - d^2$

[4–3] **19.** $2x^2 + 7x - 4$

[4–4] **20.** $y^4 - 4z^2$

[4–3] **21.** $6x^2 + 11x + 4$

[4–2] **22.** $x^2 + 3x - 28$

Find the solution set for problems 23–28 and solve problems 29 and 30.

[2–6] **23.** $2(4x - 3) = 5x - 7$

[4–7] **24.** $x^2 = 9$

[2–6] **25.** $3(x + 4) - 2(x - 3) = 12$

[2–6] **26.** $\frac{1}{4}x + 2 = \frac{1}{2}x - 1$

[6–5] **27.** $\frac{3x + 1}{9} + \frac{1}{12} = \frac{2x - 1}{3}$

[4–7] **28.** $2x^2 + 3x + 1 = 0$

[2–9] **29.** $-3 < 2x - 5 < 11$

[2–9] **30.** $3x + 5 > x + 12$

[7–3] **31.** Find the slope of the line passing through points (4,2) and (3,5).

[7–4] **32.** Write $8x - 2y = 4$ in slope-intercept form and determine the slope and y-intercept.

[8–2] **33.** Solve the system of equations.
$5x - y = 4$
$x + 3y = 2$

[2–8] **34.** One number is four times a second number and their sum is 70. Find the numbers.

[5–4] **35.** A punch machine can make 18 holes in 4 minutes. How many holes can the machine make in 5 hours?

[4–8] **36.** The product of two consecutive positive even integers is 288. Find the integers.

[2–8] **37.** The width of a rectangle is 6 feet less than its length. The perimeter of the rectangle is 96 feet. Find the dimensions.

Chapter 9

Exercise 9–1

Answers to odd-numbered problems

1. 10 **3.** 2 **5.** -12 **7.** -11 **9.** 11 **11.** -4 **13.** 4.243 **15.** 6.403 **17.** -7.211 **19.** 24 amperes **21.** 13 units **23.** 5 meters **25.** 13 inches **27.** 8 yards **29.** 20 millimeters **31.** 2 **33.** 5 **35.** -2 **37.** 3 **39.** -3 **41.** 2 **43.** -3 **45.** 1 **47.** -1 **49.** 9 units

Solutions to trial exercise problems

19. $I = \sqrt{\frac{\text{watts}}{\text{ohms}}} = \sqrt{\frac{(\ \ )}{(\ \ )}} = \sqrt{\frac{1{,}728}{3}} = \sqrt{576} = 24$ Answer: 24 amperes **31.** $\sqrt[3]{8} = 2$, since $2 \cdot 2 \cdot 2 = 2^3 = 8$. **39.** $-\sqrt[4]{81} = -3$, since $3 \cdot 3 \cdot 3 \cdot 3 = 3^4 = 81$ and the negative sign indicates that we want the negative root. **45.** $\sqrt[10]{1} = 1$, since $1^{10} = 1$. 1 raised to any power is 1 and the nth root of 1 is 1.

Review exercises

1. 3^2 **2.** $2^2 \cdot 3$ **3.** 2^3 **4.** $2^3 \cdot 5$ **5.** $2 \cdot 5^2$ **6.** 3^4 **7.** 2^6 **8.** 2^4

Exercise 9–2

Answers to odd-numbered problems

1. 4 **3.** $2\sqrt{5}$ **5.** $3\sqrt{5}$ **7.** $4\sqrt{2}$ **9.** $4\sqrt{5}$ **11.** $7\sqrt{2}$ **13.** $a^3\sqrt{a}$ **15.** $2ab\sqrt{b}$ **17.** $3ab^2\sqrt{3ab}$ **19.** $3\sqrt{2}$ **21.** 15 **23.** $2\sqrt{15}$ **25.** $5\sqrt{15}$ **27.** $5\sqrt{3}$ **29.** $5x\sqrt{3}$ **31.** $4b\sqrt{3a}$ **33.** 13 feet **35.** 24 mph **37.** $2\sqrt[5]{2}$ **39.** $2\sqrt[3]{3}$ **41.** $b^2\sqrt[3]{b^2}$ **43.** y^3 **45.** $b\sqrt[3]{4a^2}$ **47.** $2ab\sqrt[3]{2ab^2}$ **49.** $3ab^3\sqrt[3]{3a^2b^2}$ **51.** $b\sqrt[3]{b}$ **53.** a **55.** $3ab\sqrt[3]{2b}$ **57.** $3b\sqrt[4]{3a^3}$ **59.** $4a^5b^3\sqrt[3]{3b}$ **61.** 2 inches **63.** 2 inches

Solutions to trial exercise problems

15. $\sqrt{4a^2b^3} = \sqrt{4 \cdot a^2 \cdot b^2 \cdot b} = \sqrt{4}\sqrt{a^2}\sqrt{b^2}\sqrt{b} = 2ab\sqrt{b}$
19. $\sqrt{6}\sqrt{3} = \sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$
42. $\sqrt[3]{x^9} = \sqrt[3]{x^3 \cdot x^3 \cdot x^3} = \sqrt[3]{x^3}\sqrt[3]{x^3}\sqrt[3]{x^3} = x \cdot x \cdot x = x^3$
54. $\sqrt[3]{5a^2b}\ \sqrt[3]{75a^2b^2} = \sqrt[3]{375a^4b^3} = \sqrt[3]{125 \cdot 3 \cdot a^3 \cdot a \cdot b^3}$ $= \sqrt[3]{125}\sqrt[3]{a^3}\sqrt[3]{b^3}\sqrt[3]{3a} = 5ab\sqrt[3]{3a}$
61. $h = \sqrt[3]{\frac{12I}{b}} = \sqrt[3]{\frac{12(\ \ )}{(\ \ )}} = \sqrt[3]{\frac{12(2)}{(3)}} = \sqrt[3]{\frac{24}{3}} = \sqrt[3]{8}$ $= \sqrt[3]{2^3} = 2$ Answer: $h = 2$ inches

Review exercises

1. $\frac{7}{8}$ **2.** $-4x^2y$ **3.** $\frac{2y + 10}{y + 1}$ **4.** 5 **5.** 3 **6.** 4 **7.** 6 **8.** x

Exercise 9–3

Answers to odd-numbered problems

1. $\frac{3}{5}$ **3.** $\frac{5}{7}$ **5.** $\frac{\sqrt{3}}{2}$ **7.** $\frac{8}{a}$ **9.** $\frac{\sqrt{2}}{2}$ **11.** $\frac{2\sqrt{7}}{7}$ **13.** $\frac{\sqrt{15}}{15}$ **15.** $\frac{2\sqrt{3}}{15}$ **17.** $\sqrt{2}$ **19.** $\frac{5\sqrt{2}}{2}$ **21.** $\frac{x\sqrt{y}}{y}$ **23.** $\frac{\sqrt{x}}{x}$ **25.** a^2 **27.** $\sqrt{41}$ meters **29.** 12 feet **31.** $\frac{1}{2}$ **33.** $\frac{3}{5}$ **35.** $\frac{a^2\sqrt[3]{3}}{b}$ **37.** $\frac{\sqrt[5]{a^4}}{b^2}$ **39.** $\frac{ab^2\sqrt[4]{bc}}{c^3}$ **41.** $\frac{\sqrt[5]{x^3y^2}}{z^3}$ **43.** $\frac{\sqrt[3]{20}}{5}$ **45.** $\frac{2\sqrt[4]{5}}{5}$ **47.** $\frac{x\sqrt[3]{y}}{y}$ **49.** $b\sqrt[3]{a}$ **51.** $\frac{a\sqrt[3]{bc^2}}{bc}$ **53.** $\frac{\sqrt[3]{a^2bc^2}}{bc}$ **55.** $\sqrt[3]{a^2b}$ **57.** 3 units

Solutions to trial exercise problems

15. $\sqrt{\frac{4}{75}} = \frac{\sqrt{4}}{\sqrt{75}} = \frac{\sqrt{2^2}}{\sqrt{3 \cdot 5^2}} = \frac{2}{5\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{5 \cdot 3} = \frac{2\sqrt{3}}{15}$
35. $\sqrt[3]{\frac{3a^6}{b^3}} = \frac{\sqrt[3]{3a^6}}{\sqrt[3]{b^3}} = \frac{a^2\sqrt[3]{3}}{b}$ **39.** $\sqrt[4]{\frac{a^4b^9}{c^{11}}} = \frac{\sqrt[4]{a^4b^9}}{\sqrt[4]{c^{11}}}$
$= \frac{ab^2\sqrt[4]{b}}{c^2\sqrt[4]{c^3}} \cdot \frac{\sqrt[4]{c^1}}{\sqrt[4]{c^1}} = \frac{ab^2\sqrt[4]{bc}}{c^2\sqrt[4]{c^4}} = \frac{ab^2\sqrt[4]{bc}}{c^2 \cdot c} = \frac{ab^2\sqrt[4]{bc}}{c^3}$
45. $\sqrt[4]{\frac{16}{125}} = \frac{\sqrt[4]{16}}{\sqrt[4]{125}} = \frac{\sqrt[4]{2^4}}{\sqrt[4]{5^3}} = \frac{2}{\sqrt[4]{5^3}} \cdot \frac{\sqrt[4]{5^1}}{\sqrt[4]{5^1}} = \frac{2\sqrt[4]{5^1}}{\sqrt[4]{5^4}} = \frac{2\sqrt[4]{5}}{5}$
49. $\frac{ab}{\sqrt[3]{a^2}} \cdot \frac{\sqrt[3]{a^1}}{\sqrt[3]{a^1}} = \frac{ab\sqrt[3]{a}}{\sqrt[3]{a^3}} = \frac{ab\sqrt[3]{a}}{a} = b\sqrt[3]{a}$
51. $\sqrt[3]{\frac{a^3}{b^2c}} = \frac{\sqrt[3]{a^3}}{\sqrt[3]{b^2c}} = \frac{a}{\sqrt[3]{b^2c}} \cdot \frac{\sqrt[3]{b^1c^2}}{\sqrt[3]{b^1c^2}} = \frac{a\sqrt[3]{bc^2}}{\sqrt[3]{b^3c^3}} = \frac{a\sqrt[3]{bc^2}}{bc}$
57. $r = \sqrt[3]{\frac{3V}{4\pi}} = \sqrt[3]{\frac{3(\ \ )}{4(\ \ )}} = \sqrt[3]{\frac{3(113.04)}{4(3.14)}} = \sqrt[3]{\frac{339.12}{12.56}}$ $= \sqrt[3]{27} = \sqrt[3]{3^3} = 3$ Answer: 3 units

Review exercises

1. $6x$ **2.** $4y$ **3.** $8ab$ **4.** $5xy$ **5.** $x^2 - 9$ **6.** $x^2 - y^2$

Exercise 9–4

Answers to odd-numbered problems

1. $9\sqrt{3}$ **3.** $10\sqrt{5}$ **5.** $6\sqrt{3}$ **7.** $3\sqrt{7}$ **9.** $3\sqrt{a}$ **11.** $8\sqrt{a}$ **13.** $7\sqrt{xy}$ **15.** $8\sqrt{a} + 2\sqrt{ab}$ **17.** $4\sqrt{xy} + 3\sqrt{y}$ **19.** $7\sqrt{2}$ **21.** $\sqrt{3}$ **23.** $17\sqrt{7}$ **25.** $7\sqrt{2}$ **27.** $2\sqrt{3} + 8\sqrt{2}$ **29.** $7\sqrt{2a}$ **31.** $-\sqrt{x}$ **33.** $17\sqrt{2a}$ **35.** $2\sqrt{2a} + 6\sqrt{3a}$ **37.** 14 units **39.** 8.1 units by 8.1 units **41.** $6\sqrt[5]{2}$ **43.** $5\sqrt[3]{2}$ **45.** $3\sqrt[3]{3} + 10\sqrt[3]{2}$ **47.** $5\sqrt[4]{x^3}$ **49.** $\sqrt[3]{x^2y}$ **51.** $2a\sqrt[3]{b^2}$

Solutions to trial exercise problems

14. $3\sqrt{x} + 2\sqrt{y} - \sqrt{x} = 3\sqrt{x} - \sqrt{x} + 2\sqrt{y} = 2\sqrt{x} + 2\sqrt{y}$
18. $\sqrt{20} + 3\sqrt{5} = \sqrt{2^2 \cdot 5} + 3\sqrt{5} = 2\sqrt{5} + 3\sqrt{5} = 5\sqrt{5}$
29. $\sqrt{50a} + \sqrt{8a} = \sqrt{2 \cdot 5^2 \cdot a} + \sqrt{2^3 \cdot a} = 5\sqrt{2a} + 2\sqrt{2a}$ $= 7\sqrt{2a}$ **37.** $h = b + s$, where $s = 6$ units and $b = \sqrt{c^2 - s^2}$ $= \sqrt{(10)^2 - (6)^2} = \sqrt{100 - 36} = \sqrt{64} = 8$. Then $h = b + s$ $= (8) + (6) = 14$. Answer: 14 units
43. $\sqrt[3]{16} + \sqrt[3]{54} = \sqrt[3]{2^4} + \sqrt[3]{2 \cdot 3^3} = 2\sqrt[3]{2} + 3\sqrt[3]{2} = 5\sqrt[3]{2}$
50. $\sqrt[3]{x^6y} + 2x^2\sqrt[3]{y} = x^2\sqrt[3]{y} + 2x^2\sqrt[3]{y} = (1 + 2)x^2\sqrt[3]{y} = 3x^2\sqrt[3]{y}$

Review exercises

1. $6x^2 - 3xy$ **2.** $2a^4 - 2a^2b^2$ **3.** $x^2 - 2x + 1$ **4.** $y^2 - 1$ **5.** $4x^2 - 1$ **6.** $x^2 + 5xy + 6y^2$ **7.** $x^2 - 2xy + y^2$ **8.** $a^2 + 4ab + 4b^2$

Exercise 9–5

Answers to odd-numbered problems

1. $3\sqrt{2} + 3\sqrt{3}$ **3.** $\sqrt{6} + \sqrt{14}$ **5.** $6\sqrt{6} - 3\sqrt{22}$ **7.** $5\sqrt{3} - 5\sqrt{2}$ **9.** $14\sqrt{5} - 42\sqrt{2}$ **11.** $3a + \sqrt{ab}$ **13.** $14 + 7\sqrt{2}$ **15.** $12 - 25\sqrt{a} + 12a$ **17.** 1 **19.** -2 **21.** $9 + 4\sqrt{5}$ **23.** $x + 2\sqrt{xy} + y$ **25.** $x - y$ **27.** $x^2y - z$ **29.** $4x + 4y\sqrt{x} + y^2$ **31.** $11 + \sqrt{3}$ **33.** $\sqrt{a} - 3\sqrt{b}$ **35.** $\frac{-\sqrt{2} + 3}{7}$ **37.** $\frac{-14 + 7\sqrt{7}}{3}$ **39.** $\sqrt{6} + \sqrt{3}$ **41.** $\frac{6\sqrt{3} + 3\sqrt{5}}{7}$ **43.** $\frac{-3 - \sqrt{5}}{2}$ **45.** $\frac{a + 2b\sqrt{a} + b^2}{a - b^2}$

Solutions to trial exercise problems

5. $3\sqrt{2}(2\sqrt{3}-\sqrt{11}) = 3\sqrt{2}\cdot 2\sqrt{3} - 3\sqrt{2}\cdot\sqrt{11}$
$= 6\sqrt{6} - 3\sqrt{22}$ **15.** $(3-4\sqrt{a})(4-3\sqrt{a})$
$= 3\cdot 4 - 3\cdot 3\sqrt{a} - 4\sqrt{a}\cdot 4 + 4\sqrt{a}\cdot 3\sqrt{a}$
$= 12 - 9\sqrt{a} - 16\sqrt{a} + 12\cdot a = 12 - 25\sqrt{a} + 12a$
17. $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$ Conjugates, therefore
$= (\sqrt{3})^2 - (\sqrt{2})^2 = 3 - 2 = 1$ **21.** $(2+\sqrt{5})^2$
$= (2+\sqrt{5})(2+\sqrt{5}) = 2\cdot 2 + 2\sqrt{5} + 2\sqrt{5} + \sqrt{5}\sqrt{5}$
$= 4 + 4\sqrt{5} + 5 = 9 + 4\sqrt{5}$ **26.** $(2\sqrt{a}-\sqrt{b})(2\sqrt{a}+\sqrt{b})$
Conjugates, therefore $= (2\sqrt{a})^2 - (\sqrt{b})^2 = 2^2(\sqrt{a})^2 - b$
$= 4a - b$ **38.** $\dfrac{6}{3-\sqrt{6}} = \dfrac{6}{3-\sqrt{6}}\cdot\dfrac{3+\sqrt{6}}{3+\sqrt{6}} = \dfrac{6(3+\sqrt{6})}{(3)^2-(\sqrt{6})^2}$
$= \dfrac{6(3+\sqrt{6})}{9-6} = \dfrac{6(3+\sqrt{6})}{3} = 2(3+\sqrt{6}) = 6 + 2\sqrt{6}$
43. $\dfrac{1+\sqrt{5}}{1-\sqrt{5}} = \dfrac{1+\sqrt{5}}{1-\sqrt{5}}\cdot\dfrac{1+\sqrt{5}}{1+\sqrt{5}} = \dfrac{(1)^2+\sqrt{5}+\sqrt{5}+(\sqrt{5})^2}{(1)^2-(\sqrt{5})^2}$
$= \dfrac{1+2\sqrt{5}+5}{1-5} = \dfrac{6+2\sqrt{5}}{-4} = \dfrac{2(3+\sqrt{5})}{-4}$
$= \dfrac{-(3+\sqrt{5})}{2} = \dfrac{-3-\sqrt{5}}{2}$

Review exercises

1. 64 **2.** 32 **3.** 9 **4.** $\dfrac{1}{27}$ **5.** $\dfrac{1}{x^6}$ **6.** $8a^6b^3$
7. $\dfrac{y^3}{x}$ **8.** $\dfrac{1}{x^9}$

Exercise 9–6

Answers to odd-numbered problems

1. 6 **3.** a^2 **5.** 2 **7.** -3 **9.** 9 **11.** 27 **13.** $\dfrac{1}{5}$ **15.** $\dfrac{1}{8}$
17. $-\dfrac{1}{2}$ **19.** 4 **21.** $2^{5/6}$ **23.** x **25.** $c^{3/4}$ **27.** 2 **29.** $2^{1/6}$
31. $a^{3/5}$ **33.** $y^{1/6}$ **35.** $a^{1/3}$ **37.** $x^{2/3}$ **39.** $c^{1/6}$ **41.** $a^{1/3}$
43. $\dfrac{1}{x^{1/6}} = \dfrac{x^{5/6}}{x}$ **45.** $\dfrac{1}{y^{1/4}} = \dfrac{y^{3/4}}{y}$ **47.** $8a^3$ **49.** $4a^4b^2$
51. $b^{1/2}c^{1/4}$ **53.** $a^{1/2}b^{2/3}$ **55.** 81 **57.** 49 mph **59.** 24 miles
61. Take the square root 3 times, that is, the square root of the square root of the square root is the eighth root.

Solutions to trial exercise problems

3. $(a^6)^{1/3} = a^{6/1\cdot 1/3} = a^2$ **7.** $(-27)^{1/3} = \sqrt[3]{(-27)} = -3$
9. $(27)^{2/3} = (\sqrt[3]{27})^2 = (3)^2 = 9$ **17.** $(-8)^{-1/3} = \dfrac{1}{(-8)^{1/3}}$
$= \dfrac{1}{\sqrt[3]{-8}} = \dfrac{1}{-2} = -\dfrac{1}{2}$ **35.** $(a^{2/3})^{1/2} = a^{2/3\cdot 1/2} = a^{1/3}$
39. $(c^{-1/4})^{-2/3} = c^{(-1/4)\cdot(-2/3)} = c^{1/6}$ **51.** $\dfrac{b^{3/4}c^{1/2}}{b^{1/4}c^{1/4}}$
$= b^{3/4-1/4}c^{1/2-1/4} = b^{2/4}c^{2/4-1/4} = b^{1/2}c^{1/4}$

Review exercises

1. 7 **2.** x **3.** $x+1$ **4.** x^2+2x+1 **5.** x^2-4x+4
6. $\{-2,3\}$ **7.** $\{2,8\}$ **8.** $\{-1,0\}$

Exercise 9–7

Answers to odd-numbered problems

1. $\{16\}$ **3.** $\{81\}$ **5.** $\{11\}$ **7.** $\{43\}$ **9.** $\{12\}$ **11.** $\{5\}$
13. $\{9\}$ **15.** $\{36\}$ **17.** $\emptyset$ **19.** $\emptyset$ **21.** $\{4\}$ **23.** $\{4\}$ **25.** $\{3\}$
27. $\{2\}$ **29.** $\{5\}$ **31.** $\{3\}$ **33.** $\{16\}$ **35.** $\{9\}$ **37.** $\{2\}$
39. $\left\{-\dfrac{3}{4}\right\}$ **41.** $\emptyset$ **43.** $\{3\}$ **45.** $\{4\}$ **47.** $\{8\}$ **49.** $\{12\}$
51. $\{2\}$ **53.** 19 **55.** 24 **57.** 4 **59.** 5 **61.** 24 feet
63. 108 feet **65.** 100

Solutions to trial exercise problems

17.
$$\begin{aligned}\sqrt{x}+7 &= 5\\ \sqrt{x} &= -2\\ (\sqrt{x})^2 &= (-2)^2\\ x &= 4\end{aligned}$$

Check:
$$\begin{aligned}\sqrt{4}+7 &= 5\\ 2+7 &= 5\\ 9 &= 5 \text{ (false)}\end{aligned}$$

Therefore no solution and the solution set is $\emptyset$.

21.
$$\begin{aligned}\sqrt{2x+1} &= \sqrt{x+5}\\ (\sqrt{2x+1})^2 &= (\sqrt{x+5})^2\\ 2x+1 &= x+5\\ x+1 &= 5\\ x &= 4\end{aligned}$$

Check:
$$\begin{aligned}\sqrt{2(4)+1} &= \sqrt{(4)+5}\\ \sqrt{8+1} &= \sqrt{9}\\ \sqrt{9} &= \sqrt{9}\\ 3 &= 3 \text{ (true)}\end{aligned}$$

$\{4\}$

39.
$$\begin{aligned}\sqrt{x^2+1} &= x+2\\ (\sqrt{x^2+1})^2 &= (x+2)^2\\ x^2+1 &= (x+2)(x+2)\\ x^2+1 &= x^2+2x+2x+4\\ x^2+1 &= x^2+4x+4\\ 1 &= 4x+4\\ -3 &= 4x\\ -\frac{3}{4} &= x\end{aligned}$$

Check:
$$\begin{aligned}\sqrt{\left(-\frac{3}{4}\right)^2+1} &= \left(-\frac{3}{4}\right)+2\\ \sqrt{\frac{9}{16}+1} &= -\frac{3}{4}+\frac{8}{4}\\ \sqrt{\frac{9}{16}+\frac{16}{16}} &= \frac{5}{4}\\ \sqrt{\frac{25}{16}} &= \frac{5}{4}\\ \frac{\sqrt{25}}{\sqrt{16}} &= \frac{5}{4}\\ \frac{5}{4} &= \frac{5}{4} \text{ (true)}\end{aligned}$$

$\left\{-\dfrac{3}{4}\right\}$

43.
$$\begin{aligned}\sqrt{x+6} &= x\\ (\sqrt{x+6})^2 &= (x)^2\\ x+6 &= x^2\\ 0 &= x^2-x-6\\ 0 &= (x-3)(x+2)\\ x-3 &= 0 \text{ or } x+2 = 0\\ x &= 3 \text{ or } \quad x = -2\end{aligned}$$

Check:
$$\sqrt{(3)+6}=(3)$$
$$\sqrt{9}=3$$
$$3=3 \text{ (true)}$$
$$\sqrt{(-2)+6}=(-2)$$
$$\sqrt{4}=-2$$
$$2=-2 \text{ (false)}$$
$\{3\}$

47.
$$\sqrt{x-4}=x-6$$
$$(\sqrt{x-4})^2=(x-6)^2$$
$$x-4=(x-6)(x-6)$$
$$x-4=x^2-6x-6x+36$$
$$x-4=x^2-12x+36$$
$$0=x^2-13x+40$$
$$0=(x-8)(x-5)$$
$$x-8=0 \text{ or } x-5=0$$
$$x=8 \text{ or } \quad x=5$$
Check:
$$\sqrt{(8)-4}=(8)-6$$
$$\sqrt{4}=2$$
$$2=2 \text{ (true)}$$
$$\sqrt{(5)-4}=(5)-6$$
$$\sqrt{1}=-1$$
$$1=-1 \text{ (false)}$$
$\{8\}$

57. Let x = the number.
$$\sqrt{x+12}=x$$
$$(\sqrt{x+12})^2=(x)^2$$
$$x+12=x^2$$
$$0=x^2-x-12$$
$$0=(x-4)(x+3)$$
$$x-4=0 \text{ or } x+3=0$$
$$x=4 \text{ or } \quad x=-3$$
Check:
$$\sqrt{(4)+12}=(4)$$
$$\sqrt{16}=4$$
$$4=4 \text{ (true)}$$
$$\sqrt{(-3)+12}=(-3)$$
$$\sqrt{9}=-3$$
$$3=-3 \text{ (false)}$$
Hence the number is 4.

Review exercises

1. $(x+2)(x-2)$ **2.** $(x+3)(x+6)$ **3.** $(x+2)(x-5)$ **4.** $(x-3)^2$ **5.** 9 **6.** 7 **7.** 11 **8.** $\{-8,8\}$

Chapter 9 review

1. 9 **2.** 5 **3.** -3 **4.** -7 **5.** $2\sqrt{10}$ **6.** $3ab\sqrt{2b}$ **7.** $2\sqrt{7}$ **8.** $6\sqrt{5}$ **9.** $\frac{4\sqrt{17}}{17}$ **10.** $\frac{\sqrt{14}}{6}$ **11.** $\frac{\sqrt{ab}}{b}$ **12.** $\frac{\sqrt{xy}}{y^2}$ **13.** $\frac{\sqrt{ab}}{b}$ **14.** $\frac{2\sqrt{xy}}{y}$ **15.** $7\sqrt{7}$ **16.** $8\sqrt{2}$ **17.** $3\sqrt{5}$ **18.** $24\sqrt{3}$ **19.** $\sqrt{2a}$ **20.** $17\sqrt{x}$ **21.** $\sqrt{15}-\sqrt{21}$ **22.** $2\sqrt{35}+2\sqrt{15}$ **23.** $8-2\sqrt{7}$ **24.** $39-12\sqrt{3}$ **25.** $8+2\sqrt{15}$ **26.** $4a-b$ **27.** $-\sqrt{3}-2$ **28.** $\frac{-\sqrt{6}+4}{5}$ **29.** $\frac{\sqrt{a}-b}{a-b^2}$ **30.** $\frac{\sqrt{xy}-x}{y-x}$ **31.** $\frac{a^2-a\sqrt{b}}{a^2-b}$ **32.** $\frac{-11-6\sqrt{2}}{7}$ **33.** 6 **34.** 4 **35.** -2 **36.** $-\frac{1}{4}$ **37.** a **38.** $b^{13/12}$ **39.** $a^{1/4}$ **40.** $a^{9/8}$ **41.** $8a^3b^6$ **42.** $a^{3/2}b^{1/2}$ **43.** $\{64\}$ **44.** $\{53\}$ **45.** $\{4\}$ **46.** $\{1\}$ **47.** $\{2\}$ **48.** $\{3\}$ **49.** $\{-2\}$ **50.** $\{4,3\}$

Chapter 9 cumulative test

1. 45 **2.** x^7 **3.** x^6 **4.** $2x^2+2x+13$ **5.** $8a^4b^3-12a^3b^4+16a^2b^5$ **6.** $2a^3b^2$ **7.** -36 **8.** $9a^2-6ab+b^2$ **9.** $\frac{a-3}{3a+6}$ **10.** $16\sqrt{3}$ **11.** -2 **12.** $\frac{x+3}{x-1}$ **13.** $\frac{x\sqrt{x}+\sqrt{xy}}{x^2-y}$ **14.** $3xy^2\sqrt[3]{3xz}$ **15.** $25x^2-y^2$ **16.** $2x-4y$ **17.** $2a^3b^3(3b-b^2+4a^2)$ **18.** $(5c+d)(5c-d)$ **19.** $(2x-1)(x+4)$ **20.** $(y^2+2z)(y^2-2z)$ **21.** $(2x+1)(3x+4)$ **22.** $(x+7)(x-4)$ **23.** $\left\{-\frac{1}{3}\right\}$ **24.** $\{-3,3\}$ **25.** $\{-6\}$ **26.** $\{12\}$ **27.** $\left\{\frac{19}{12}\right\}$ **28.** $\left\{-1,-\frac{1}{2}\right\}$ **29.** $1<x<8$ **30.** $x>\frac{7}{2}$ **31.** -3 **32.** $y=4x-2$; slope is 4; y-intercept is $(0,-2)$ **33.** $\left(\frac{7}{8},\frac{3}{8}\right)$ **34.** 14, 56 **35.** 1,350 **36.** 16, 18 **37.** 21 feet by 27 feet

Chapter 10

Exercise 10–1

Answers to odd-numbered problems

1. $\{-5,3\}$ **3.** $\left\{-\frac{3}{2},2\right\}$ **5.** $\{-2,2\}$ **7.** $\{-8,8\}$ **9.** $\{-\sqrt{11},\sqrt{11}\}$ **11.** $\{2\sqrt{5},-2\sqrt{5}\}$ **13.** $\{-\sqrt{3},\sqrt{3}\}$ **15.** $\{-4\sqrt{2},4\sqrt{2}\}$ **17.** $\{-3,3\}$ **19.** $\{-\sqrt{6},\sqrt{6}\}$ **21.** $\{-5\sqrt{2},5\sqrt{2}\}$ **23.** $\{-2\sqrt{2},2\sqrt{2}\}$ **25.** $\{-2\sqrt{2},2\sqrt{2}\}$ **27.** $\{-\sqrt{2},\sqrt{2}\}$ **29.** $\left\{-\frac{\sqrt{6}}{5},\frac{\sqrt{6}}{5}\right\}$ **31.** $\{-\sqrt{11},\sqrt{11}\}$ **33.** $\{-4,0\}$ **35.** $\{-1,9\}$ **37.** $\{-3-\sqrt{6},-3+\sqrt{6}\}$ **39.** $\{9-3\sqrt{2},9+3\sqrt{2}\}$ **41.** $\{-5+4\sqrt{2},-5-4\sqrt{2}\}$ **43.** $\{-a-6,-a+6\}$ **45.** $\{6-a,6+a\}$ **47.** $\{p-q,p+q\}$ **49.** $\left\{-\frac{1}{2},\frac{7}{2}\right\}$ **51.** 5 meters **53.** 2 feet **55.** 9, -9 **57.** 0,9 **59.** 7 inches, 14 inches **61.** length = 24 meters; width = 6 meters **63.** 4 and 8 **65.** $5\sqrt{2}$ centimeters

Solutions to trial exercise problems

11. $a^2=20$

Extract the roots.
$$a=\sqrt{20} \text{ or } a=-\sqrt{20}$$
$$a=\sqrt{4\cdot 5} \text{ or } a=-\sqrt{4\cdot 5}$$
$$a=2\sqrt{5} \text{ or } a=-2\sqrt{5}$$
$\{2\sqrt{5},-2\sqrt{5}\}$

18. $5x^2=75$

Divide each member by 5.
$$x^2=15$$
Extract the roots.
$$x=\sqrt{15} \text{ or } x=-\sqrt{15}$$
$\{\sqrt{15},-\sqrt{15}\}$

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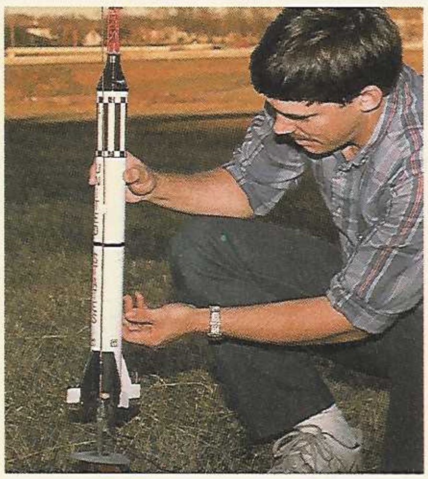

Chapter 5 ■ Rational Expressions, Ratio and Proportion

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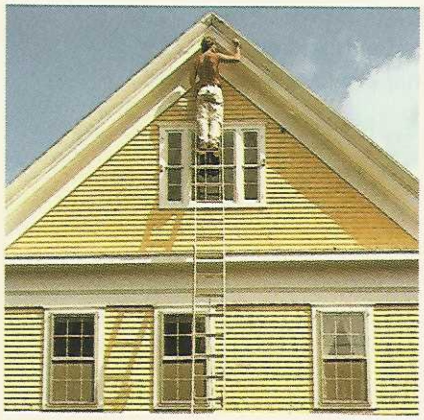
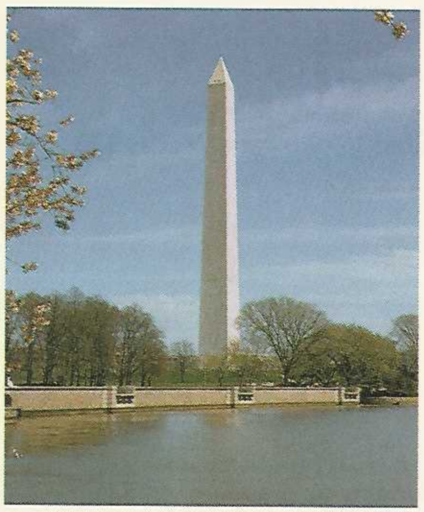


Chapter 10 ■ Solutions of Quadratic Equations



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